

REPRESENTATION THEORY OF REDUCTIVE p -ADIC GROUPS

YANG HU

1. ALGEBRAIC GROUPS

Throughout this section, k is algebraically closed of characteristic zero, and algebraic groups are linear algebraic groups over k . Unless otherwise stated, G is connected. We write G^0 for the identity component and $Z_G(H)$ and $N_G(H)$ for the centralizer and normalizer of a subgroup H . The basic structure results used here are treated in [Spr].

1.1. The G -action. Let $\rho : G \rightarrow \mathrm{GL}(V)$ be a rational representation on a finite-dimensional vector space. Define

$$\mathbb{P}(V) = (V \setminus \{0\})/k^\times = \{\text{one-dimensional subspaces of } V\}.$$

The induced action is $g \cdot [v] = [\rho(g)v]$. Thus $[v]$ is fixed by G exactly when kv is G -stable; this does not require v itself to be fixed. If $\dim V = n$, then $\mathbb{P}(V) \simeq \mathbb{P}^{n-1}$.

For a G -variety X and $x \in X$, write

$$G_x = \{g \in G : g \cdot x = x\}, \quad G \cdot x = \{g \cdot x : g \in G\}.$$

The stabilizer is closed, the orbit is locally closed, and the orbit map identifies G/G_x with $G \cdot x$. In particular, $\dim(G \cdot x) = \dim G - \dim G_x$. Every nonempty G -variety contains a closed orbit: choose an orbit of minimum dimension and use the fact that every orbit in its boundary has smaller dimension.

1.2. Parabolic subgroups and Borel subgroups.

Definition 1.1. A **homogeneous space** for G is a nonempty algebraic variety X equipped with a transitive algebraic G -action. Choosing $x \in X$ identifies X with G/G_x .

Lemma 1.2. *Let X and Y be homogeneous spaces for G . A bijective G -morphism $\phi : X \rightarrow Y$ is an isomorphism. Consequently, X is complete if and only if Y is complete.*

Proof. Choose $x \in X$ and put $y = \phi(x)$. Equivariance gives $G_x \subseteq G_y$, and injectivity gives equality. Under the identifications with the corresponding quotients, ϕ is the identity map on G/G_x . Here characteristic zero ensures that the stabilizers are smooth, so equality of their k -points gives equality of the closed subgroups. $\square$

Definition 1.3. A closed subgroup $P \subseteq G$ is *parabolic* if G/P is complete, that is, proper over k .

Lemma 1.4. *If P is parabolic, then G/P is projective.*

Proof. The quotient of a linear algebraic group by a closed subgroup is quasi-projective. Choose a locally closed immersion $G/P \hookrightarrow \mathbb{P}^N$. Since G/P is complete, this immersion is proper, hence a closed immersion. The converse follows because projective varieties are complete. $\square$

Lemma 1.5. *If P is parabolic in G and Q is parabolic in P , then Q is parabolic in G .*

Proof. The map $G/Q \rightarrow G/P$ has fiber P/Q . After base change by the faithfully flat quotient map $G \rightarrow G/P$, it becomes the projection $G \times (P/Q) \rightarrow G$. This projection is proper, so properness descends to $G/Q \rightarrow G/P$. Its composite with $G/P \rightarrow \mathrm{Spec} k$ is therefore proper. $\square$

Lemma 1.6. (1) *If P is parabolic in G and $P \subseteq Q \subseteq G$ with Q closed, then Q is parabolic.*

(2) *Allowing G to be disconnected, P is parabolic in G if and only if P^0 is parabolic in G^0 .*

Proof. For (1), the surjection $G/P \rightarrow G/Q$ has complete source, so its image G/Q is complete. For (2), G/P is a finite disjoint union of translates of $G^0/(P \cap G^0)$, and

$$G^0/P^0 \rightarrow G^0/(P \cap G^0)$$

is finite and surjective. Completeness is equivalent on the two sides of a finite surjective morphism, which proves the assertion. $\square$

Proposition 1.7. *A connected group G contains a proper parabolic subgroup if and only if G is not solvable.*

Proof. First suppose that G is solvable, and let P be parabolic. We use induction on $\dim G$. Put $D = [G, G]$; it is a closed connected normal subgroup, and $\dim D < \dim G$ if $G \neq 1$. The subgroup PD is closed and normal because it is the inverse image of the image of P in the commutative quotient G/D . Thus G/PD is affine. It is also a complete connected image of G/P , so it is a point. Hence $G = PD$, and

$$D/(D \cap P) \simeq G/P.$$

By induction, $D \cap P = D$, whence $P = G$.

Conversely, embed G faithfully in $\mathrm{GL}(V)$ and let it act on the complete flag variety of V . A closed orbit has the form G/P and is projective, so P is parabolic. If $P = G$, then G stabilizes a complete flag and is conjugate into the upper triangular group, hence is solvable. Thus a nonsolvable G gives $P \neq G$. $\square$

Theorem 1.8 (Borel fixed point theorem). *A connected solvable linear algebraic group acting on a nonempty complete variety X has a fixed point.*

Proof. Choose a closed orbit $G \cdot x \subseteq X$. It is complete, so G_x is parabolic. Proposition 1.7 gives $G_x = G$, and therefore x is fixed. $\square$

Definition 1.9. A **Borel subgroup** of G is a maximal connected closed solvable subgroup.

Theorem 1.10. (1) *A closed subgroup of G is parabolic if and only if it contains a Borel subgroup.*
 (2) *Every Borel subgroup is parabolic.*
 (3) *Any two Borel subgroups are conjugate.*

Proof. In a faithful representation, choose a closed G -orbit in the complete flag variety and let P be its stabilizer. The subgroup P is solvable, since it preserves a complete flag. By Lemma 1.6, $B = P^0$ is solvable and parabolic. If a connected solvable subgroup S contains B , then S/B is a closed fiber of $G/B \rightarrow G/S$, hence is complete. Proposition 1.7 gives $S = B$, so B is Borel.

Any Borel subgroup B' acts on the complete variety G/B . By Theorem 1.8, it fixes some gB , so $B' \subseteq gBg^{-1}$. Maximality gives equality. This proves (2) and (3).

If P is parabolic, a Borel subgroup acting on G/P has a fixed point, so P contains a conjugate Borel. Conversely, if $B \subseteq P$, the surjection $G/B \rightarrow G/P$ proves completeness. $\square$

Corollary 1.11. *Let $\phi : G \rightarrow G'$ be a surjective homomorphism of connected linear algebraic groups. If P is parabolic (respectively Borel), then $\phi(P)$ is parabolic (respectively Borel).*

Proof. The image is closed, and $G/P \rightarrow G'/\phi(P)$ is surjective, proving the parabolic assertion. If $P = B$ is Borel, then $\phi(B)$ is connected and solvable. Choose a Borel B' containing it. The quotient $B'/\phi(B)$ is complete, being a closed fiber of $G'/\phi(B) \rightarrow G'/B'$. A connected solvable group has no proper parabolic, so $B' = \phi(B)$. $\square$

Corollary 1.12. *If B is a Borel subgroup of G , then $Z_G(B) = Z(G)$ and $Z(B) = Z(G)$.*

Proof. For $c \in Z_G(B)$, the morphism $g \mapsto cgc^{-1}g^{-1}$ is constant on right B -cosets and descends to $G/B \rightarrow G$. Since G/B is complete and connected and G is affine, this map is constant, with value 1. Thus $c \in Z(G)$, proving $Z_G(B) = Z(G)$. The self-normalizing property $N_G(B) = B$, stated below, gives $Z(G) \subseteq B$, and hence $Z(B) = Z(G)$. $\square$

Corollary 1.13. *If a Borel subgroup B is nilpotent, then $G = B$.*

Proof. Induct on $\dim B$. If $B = 1$, then $G = G/B$ is affine and complete, hence trivial. Otherwise a connected nilpotent group of positive dimension has a positive-dimensional center. Put $C = Z(B)^0 \subseteq Z(G)$. The image B/C is a nilpotent Borel subgroup of G/C . By induction $G/C = B/C$, so $G = B$. $\square$

Exercise 1.14. For $G = \mathrm{GL}_2(k)$, identify G/B with $\mathbb{P}^1$ by sending gB to the line spanned by the first column of g . Verify that the stabilizer of $[1 : 0]$ is the upper triangular subgroup.

1.2.1. *The case $\mathrm{GL}_n(k)$.* Let B_n be the invertible upper triangular matrices, T_n the diagonal matrices, and U_n the upper unitriangular matrices. Then

$$B_n = T_n \ltimes U_n.$$

The quotient GL_n/B_n parametrizes complete flags $0 \subset V_1 \subset \cdots \subset V_n = k^n$, with $\dim V_i = i$. It is a closed incidence subvariety of $\prod_{i=1}^{n-1} \mathrm{Gr}(i, n)$, hence is projective.

More generally, for an ordered composition $n = n_1 + \cdots + n_r$, the standard parabolic $P_{(n_1, \dots, n_r)}$ consists of invertible block upper triangular matrices with these block sizes. Its quotient parametrizes partial flags whose nonzero proper dimensions are $n_1, n_1 + n_2, \dots, n_1 + \cdots + n_{r-1}$. Every parabolic is conjugate to one of these. Its block diagonal subgroup is a Levi subgroup, isomorphic to $\prod_i \mathrm{GL}_{n_i}$, and its unipotent radical consists of matrices with identity diagonal blocks.

1.3. **Connectedness and Solvability.** In this subsection G is connected and solvable. Recall that the multiplicative Jordan decomposition writes every $g \in G$ uniquely as $g = g_s g_u = g_u g_s$, where g_s is semisimple and g_u is unipotent. Both factors lie in G , and homomorphisms of algebraic groups preserve this decomposition.

Theorem 1.15 (Lie–Kolchin). *If $G \subseteq \mathrm{GL}_n(k)$, then there is an $x \in \mathrm{GL}_n(k)$ such that xGx^{-1} is contained in the upper triangular group.*

Proof. Apply Theorem 1.8 to $\mathbb{P}^{n-1}$ to obtain a G -stable line L . Apply the same argument to the induced representation on k^n/L , and continue by induction. This yields a G -stable complete flag; a basis adapted to it triangularizes G . $\square$

Corollary 1.16. *Suppose further that G is nilpotent. Then the sets G_s and G_u of semisimple and unipotent elements are closed connected subgroups, G_s is a central torus, and multiplication gives an isomorphism*

$$G_s \times G_u \simeq G$$

of algebraic groups.

Proof. This is the Jordan decomposition theorem for connected nilpotent algebraic groups; see [Spr, Chapter 6]. In this case the semisimple elements form a central torus, so the two factors commute and their intersection is trivial. The asserted product is the corresponding algebraic-group decomposition. For a general solvable group, G_s need not be a subgroup. $\square$

Corollary 1.17. (1) *The derived subgroup $[G, G]$ is closed, connected, normal, and unipotent.*

(2) *The set G_u is a closed connected nilpotent normal subgroup, and G/G_u is a torus.*

Proof. Triangularize G and map it to the diagonal torus. Its kernel consists exactly of the unipotent elements. A closed subgroup of a unipotent group in characteristic zero is connected and nilpotent. The image is a connected closed subgroup of a torus, hence a torus. The derived subgroup lies in the kernel; its closedness and connectedness are the standard derived-subgroup theorem for connected algebraic groups. $\square$

Lemma 1.18. *If G is not a torus, then it has a closed normal subgroup $N \simeq \mathbb{G}_a$ contained in $Z(G_u)$.*

Proof. Put $U = G_u \neq 1$. The nonzero vector space $\mathrm{Lie}(Z(U))$ is stable under conjugation by G . Lie–Kolchin provides a G -stable line in it. In characteristic zero, the exponential map identifies a unipotent group with its nilpotent Lie algebra as varieties; the exponential of this central line is the required normal subgroup N . $\square$

Definition 1.19. A torus is an algebraic group isomorphic to $(\mathbb{G}_m)^r$ for some $r \geq 0$. In a connected solvable G , a *maximal torus* is a subtorus T satisfying $\dim T = \dim(G/G_u)$. The structure theorem below shows that this agrees with maximality under inclusion.

Theorem 1.20. (1) *Every semisimple element of G lies in a maximal torus.*

(2) *The centralizer $Z_G(s)$ of a semisimple element is connected.*

(3) *Maximal tori exist, and any two are conjugate by an element of G_u .*

(4) *If T is a maximal torus, then $G = T \ltimes G_u$; in particular, multiplication $T \times G_u \rightarrow G$ is an isomorphism of varieties.*

Proof. The splitting and conjugacy theorem for connected solvable algebraic groups gives a torus complement to G_u , unique up to G_u -conjugacy; it also puts every diagonalizable subgroup into such a complement [Spr, Chapter 6]. The Zariski closure of the group generated by a semisimple element is diagonalizable, proving (1). The restriction $T \rightarrow G/G_u$ is an isomorphism, which proves (3) and (4). After conjugation, assume $s \in T$. Then

$$Z_G(s) = T \ltimes Z_{G_u}(s).$$

The exponential map identifies $Z_{G_u}(s)$ with the fixed subspace of $\text{Ad}(s)$ on $\text{Lie}(G_u)$, so it is connected. $\square$

Corollary 1.21. *Let $H \subseteq G(k)$ be a subgroup all of whose elements are semisimple. Then H is contained in a maximal torus, and $Z_G(H)$ is connected and equals $N_G(H)$.*

Proof. Since $[G, G] \subseteq G_u$ and $H \cap G_u = 1$, the subgroup H is commutative. Commuting semisimple matrices are simultaneously diagonalizable, so the Zariski closure of H is diagonalizable and is contained in a maximal torus by the preceding structure theorem. Thus $Z_G(H) = T \ltimes Z_{G_u}(H)$ after choosing such a torus. The second factor is connected by the exponential-map argument with a simultaneous fixed subspace. Finally, if g normalizes H , then for $h \in H$ the commutator $ghg^{-1}h^{-1}$ lies in both H and G_u , and is therefore 1. $\square$

1.4. Maximal tori. In this subsection, G is a connected linear algebraic group.

Definition 1.22. A *maximal torus* of G is a torus not properly contained in another torus of G . A *Cartan subgroup* is a subgroup of the form $Z_G(T)^0$, where T is a maximal torus.

Theorem 1.23. *Any two maximal tori of G are conjugate.*

Proof. Every torus is connected and solvable, hence is contained in a Borel subgroup. Conjugacy of Borel subgroups reduces the assertion to conjugacy of maximal tori in a connected solvable group, proved in Theorem 1.20. $\square$

Theorem 1.24. *Let S be a torus in G . Then $Z_G(S)$ is connected. If B is a Borel subgroup of G containing S , then $Z_G(S) \cap B$ is a Borel subgroup of $Z_G(S)$. Every Borel subgroup of $Z_G(S)$ arises in this way.*

This is the centralizer theorem for tori; see [Mil, Chapter 17] or [Spr]. For the last assertion, fix one such B . Every Borel subgroup of $Z_G(S)$ is $c(B \cap Z_G(S))c^{-1}$ for some $c \in Z_G(S)$, and this equals $cBc^{-1} \cap Z_G(S)$.

Proposition 1.25. *Let T be a maximal torus and $C = Z_G(T)$. Then C is connected and nilpotent, and T is its unique maximal torus. There exists $t \in T$ contained in only finitely many conjugates of C .*

Proof. Connectedness follows from Theorem 1.24. The Cartan subgroup theorem says that C is nilpotent; see [Spr]. A connected nilpotent linear algebraic group is the direct product of its central maximal torus and its unipotent subgroup. Since T is maximal in G , it is this unique torus of C .

Choose $t \in T$ with $Z_G(t) = Z_G(T)$, as in the following lemma. If a conjugate C' of C contains t , then t is central in C' , because semisimple elements of a connected nilpotent group lie in its central torus. Thus $C' \subseteq Z_G(t) = C$. Since the two connected groups have the same dimension, $C' = C$. In fact, this choice of t lies in just one conjugate of C . $\square$

Lemma 1.26. *If S is a torus in G , there exists $s \in S$ such that $Z_G(s) = Z_G(S)$.*

Proof. Choose a faithful representation $G \hookrightarrow \mathrm{GL}(V)$ and write $V = \bigoplus_{\chi} V_{\chi}$ for its S -weight decomposition. Choose s outside the finitely many proper closed subsets $\ker(\chi\psi^{-1})$, for distinct weights χ, ψ . The eigenspaces of s are then exactly the spaces V_{χ} . Consequently an element of G commutes with s if and only if it preserves every V_{χ} , which is equivalent to commuting with S . $\square$

Theorem 1.27 (Normalizer theorem). *If B is a Borel subgroup of G , then $N_G(B) = B$. More generally, every parabolic subgroup P is connected and satisfies $N_G(P) = P$.*

This is the normalizer theorem for connected linear algebraic groups; see [Mil, Chapter 17]. Connectedness of G is essential here.

1.5. General theory of root systems. Let T be a maximal torus of G . Write

$$X = X^*(T) = \mathrm{Hom}(T, \mathbb{G}_m), \quad X^{\vee} = X_*(T) = \mathrm{Hom}(\mathbb{G}_m, T).$$

These are free abelian groups of rank $\dim T$, with the perfect pairing determined by

$$\chi(\lambda(z)) = z^{\langle \chi, \lambda \rangle}.$$

A rational representation $r : T \rightarrow \mathrm{GL}(V)$ has a weight decomposition

$$V = \bigoplus_{\chi \in X} V_{\chi}, \quad V_{\chi} = \{v : r(t)v = \chi(t)v \text{ for every } t \in T\}.$$

Only finitely many weights occur when V is finite-dimensional. Applying this to the adjoint representation gives

$$\mathfrak{g} = \mathfrak{c} \oplus \bigoplus_{\alpha \in P} \mathfrak{g}_{\alpha}, \quad \mathfrak{c} = \mathrm{Lie} Z_G(T),$$

where P is the set of nonzero weights and $\mathrm{Ad}(t)Z = \alpha(t)Z$ for $Z \in \mathfrak{g}_{\alpha}$. For a general connected group, the zero-weight space need not equal $\mathfrak{t}$.

Lemma 1.28. *For a subtorus $S \subset T$, one has $Z_G(S) = Z_G(T)$ if and only if S is not contained in $\ker \alpha$ for any $\alpha \in P$.*

Proof. The Lie algebra of $Z_G(S)$ is the S -fixed subspace of $\mathfrak{g}$:

$$\mathrm{Lie} Z_G(S) = \mathfrak{c} \oplus \bigoplus_{\substack{\alpha \in P \\ \alpha|_S = 1}} \mathfrak{g}_{\alpha}.$$

Both centralizers are connected, and $Z_G(T) \subset Z_G(S)$. The asserted equality is therefore equivalent to equality of their Lie algebras. $\square$

Definition 1.29. The *radical* $R(G)$ is the largest connected closed normal solvable subgroup of G . The *unipotent radical* $R_u(G)$ is the largest connected closed normal unipotent subgroup. One has $R_u(G) = R_u(R(G))$. The group G is *reductive* if $R_u(G) = 1$, and *semisimple* if $R(G) = 1$.

Definition 1.30. The *Weyl group* of (G, T) is

$$W(G, T) = N_G(T)/Z_G(T).$$

It acts on X by $(w\chi)(t) = \chi(n^{-1}tn)$, where n represents w . The Weyl group is finite.

For the root-system statements below, assume G is reductive. Then $Z_G(T) = T$, and the nonzero adjoint weights form a reduced root system R in their real span inside $X \otimes_{\mathbb{Z}} \mathbb{R}$. In particular,

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_{\alpha}.$$

The reductivity assumption matters: the adjoint weights of a nonreductive group need not form a reduced root system. We also use the fact that the subgroup generated by connected closed algebraic subgroups is closed and connected. In characteristic zero, such a subgroup is all of G as soon as its Lie algebra is $\mathfrak{g}$.

Lemma 1.31. *For $\alpha \in R$, put $G_\alpha = Z_G((\ker \alpha)^0)$. This is a connected reductive subgroup of semisimple rank one, with root system $\{\alpha, -\alpha\}$ relative to T . The groups G_α , together with T , generate G .*

Proof. The centralizer of a torus in a reductive group is reductive. Its roots are those roots of G trivial on $(\ker \alpha)^0$, namely the roots proportional to α . Reducedness gives precisely α and $-\alpha$. The Lie algebra of the subgroup generated by T and all the G_α contains $\mathfrak{t}$ and every root space, hence equals $\mathfrak{g}$. Since G is connected, the generated subgroup is G . $\square$

If R is empty, this statement reads $G = T$.

Let $V = X \otimes_{\mathbb{Z}} \mathbb{R}$. Averaging a positive definite symmetric form f gives a W -invariant form

$$(x, y) = \sum_{w \in W} f(wx, wy).$$

The nontrivial Weyl element of G_α acts as the reflection

$$s_\alpha(x) = x - 2 \frac{(x, \alpha)}{(\alpha, \alpha)} \alpha = x - \langle x, \alpha^\vee \rangle \alpha,$$

where $\alpha^\vee \in X^\vee$ is the *coroot* and $\langle \alpha, \alpha^\vee \rangle = 2$. If $G_\alpha = G_\beta$, then $\beta = \pm \alpha$, so $s_\alpha = s_\beta$.

Theorem 1.32. *The Weyl group is generated by the reflections s_α , $\alpha \in R$.*

This follows from the rank-one generation theorem for reductive groups; see [Spr]. The reflections attached to the simple roots already generate W .

1.5.1. Semisimple groups of rank 1.

Definition 1.33. The *rank* of a connected linear algebraic group is the dimension of a maximal torus. Its *semisimple rank* is the rank of $G/R(G)$. For reductive G , this equals the rank of its derived group G^{der} .

Theorem 1.34. *A connected semisimple group of rank one over k is isomorphic to SL_2 or PGL_2 .*

This is the rank-one classification theorem. Over our algebraically closed field, $\text{PGL}_2 \simeq \text{PSL}_2$. In either case, the root system is $\{\alpha, -\alpha\}$, the Weyl group has order two, and $G/B \simeq \mathbb{P}^1$.

Example 1.35. For $G = \text{SL}_2$ and $T = \{\text{diag}(a, a^{-1})\}$,

$$\alpha(\text{diag}(a, a^{-1})) = a^2, \quad \alpha^\vee(z) = \text{diag}(z, z^{-1}).$$

The matrix $n = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ represents the nontrivial Weyl element and satisfies $n^2 = \alpha^\vee(-1)$.

Proposition 1.36. *If G is connected, non-solvable and of rank one, then $|W(G, T)| = 2$ and $\dim G/B = 1$ for every Borel subgroup B .*

Proof. The semisimple quotient $G/R(G)$ has rank one. The radical lies in every Borel subgroup, and passage to this quotient preserves the Weyl group and the flag variety. Apply the rank-one classification. $\square$

1.5.2. Root datum.

Definition 1.37. A *reduced root datum* is a quadruple $(X, R, X^\vee, R^\vee)$ consisting of dual finite-rank free abelian groups and finite subsets $R \subset X \setminus \{0\}$, $R^\vee \subset X^\vee \setminus \{0\}$, with a bijection $\alpha \mapsto \alpha^\vee$, such that

$$\langle \alpha, \alpha^\vee \rangle = 2,$$

the maps

$$s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha, \quad s_\alpha^\vee(y) = y - \langle \alpha, y \rangle \alpha^\vee$$

preserve R and $R^\vee$, respectively, and $2\alpha \notin R$ for every $\alpha \in R$. The bijection is compatible with the reflections: $(s_\alpha \beta)^\vee = s_\alpha^\vee(\beta^\vee)$.

Definition 1.38. A *system of positive roots* is a set

$$R^+ = \{\alpha \in R : \langle \alpha, \lambda \rangle > 0\},$$

where $\lambda \in X^\vee$ is chosen so that $\langle \alpha, \lambda \rangle \neq 0$ for every root. Equivalently, for a W -invariant positive definite form, one may take $R^+ = \{\alpha : (\alpha, x) > 0\}$ for a vector x outside all root hyperplanes.

Remark 1.39. The *simple roots* are the positive roots not expressible as a sum of two positive roots. They form a basis of the real span of R , and every positive root is a nonnegative integral combination of them. The root datum records the character and cocharacter lattices as well as the roots; this distinguishes, for example, SL_2 from PGL_2 .

1.6. Structure of reductive algebraic groups.

1.6.1. Reductive groups of semisimple rank 1.

Proposition 1.40. *Let G be connected and reductive. Then $R(G) = Z(G)^0$ is a central torus, and $R(G) \cap G^{\mathrm{der}}$ is finite.*

Proof. The unipotent radical of $R(G)$ is characteristic in $R(G)$, hence normal in G , so it is trivial. A connected solvable group with trivial unipotent radical is a torus. A normal torus in a connected group is central: conjugation acts through its discrete automorphism group. Thus $R(G) \subset Z(G)^0$, and the reverse inclusion follows from the definition of the radical.

For finiteness, choose a faithful representation $V = \bigoplus_\chi V_\chi$ for the central torus $R(G)$. Each weight space is G -stable. If $z \in R(G) \cap G^{\mathrm{der}}$, then

$$\chi(z)^{\dim V_\chi} = \det(z|_{V_\chi}) = 1,$$

because every determinant character is trivial on G^{der} . There are finitely many possibilities for the tuple $(\chi(z))_\chi$, which determines z by faithfulness. $\square$

Now suppose that G has semisimple rank one. Fix a Borel subgroup B containing T , with positive root α .

Lemma 1.41. *There is an isomorphism $u_\alpha : \mathbb{G}_a \rightarrow U_\alpha$ onto a one-dimensional closed subgroup such that*

$$tu_\alpha(x)t^{-1} = u_\alpha(\alpha(t)x), \quad \mathrm{Im}(du_\alpha) = \mathfrak{g}_\alpha.$$

Any other such parametrization is $u'_\alpha(x) = u_\alpha(ax)$ for a unique $a \in k^\times$. Moreover, $B = T \ltimes U_\alpha$ and $\mathfrak{b} = \mathfrak{t} \oplus \mathfrak{g}_\alpha$.

Proof. By the standard structure theorem for reductive groups, G^{der} is semisimple of rank one. Its simply connected cover gives a central isogeny $\varphi_\alpha : \mathrm{SL}_2 \rightarrow G^{\mathrm{der}}$ compatible with T and B . The upper unipotent subgroup maps isomorphically onto U_α ; its matrix formulas give the equivariance and Lie algebra statements. Automorphisms of $\mathbb{G}_a$ in characteristic zero are the maps $x \mapsto ax$. $\square$

Put

$$\alpha^\vee(z) = \varphi_\alpha \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}, \quad n_\alpha = \varphi_\alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Lemma 1.42. *The coroot belongs to $X^\vee$, its image is the maximal torus $T_1 = (T \cap G^{\mathrm{der}})^0$ of G^{der} , and $n_\alpha^2 = \alpha^\vee(-1)$.*

Proof. These identities follow directly from the displayed matrices and the central isogeny φ_α . $\square$

Proposition 1.43. *Suppose $f \in k[G]$ satisfies $f(gb) = \chi(b)f(g)$ for $b \in B$, where χ is an algebraic character of B . If $f|_{G^{\mathrm{der}}}$ is nonconstant, then $\langle \chi, \alpha^\vee \rangle > 0$.*

Proof. Pull back f along φ_α . Right invariance under the upper unipotent subgroup makes this a polynomial in the first-column coordinates a, c of SL_2 ; indeed $k[\mathrm{SL}_2]^U = k[a, c]$. Right multiplication by $\mathrm{diag}(z, z^{-1})$ scales both coordinates by z . Thus the pullback is homogeneous of degree $\langle \chi, \alpha^\vee \rangle$. Since it is nonconstant, this degree is positive. $\square$

1.6.2. *Root systems of reductive groups.* Return to a connected reductive group of arbitrary rank, with root datum $(X, R, X^\vee, R^\vee)$ relative to T .

Proposition 1.44. *For each $\alpha \in R$ there is a unique root subgroup U_α admitting a parametrization $u_\alpha : \mathbb{G}_a \xrightarrow{\sim} U_\alpha$ with*

$$tu_\alpha(x)t^{-1} = u_\alpha(\alpha(t)x), \quad \text{Lie } U_\alpha = \mathfrak{g}_\alpha.$$

The torus T and all the root subgroups generate G . Each root space has dimension one.

Proof. Apply the preceding rank-one construction to $G_\alpha = Z_G((\ker \alpha)^0)$. The root-subgroup theorem gives uniqueness inside G ; see [Spr]. Generation follows because the generated connected subgroup has Lie algebra containing $\mathfrak{t}$ and every $\mathfrak{g}_\alpha$. $\square$

Fix a Borel subgroup B containing T and put $U = R_u(B)$.

Proposition 1.45. *The roots for which $U_\alpha \subset B$ form a positive system $R^+(B)$. If $R^+(B) = \{\alpha_1, \dots, \alpha_m\}$ in any fixed order, then multiplication gives an isomorphism of varieties*

$$\mathbb{A}^m \longrightarrow U, \quad (x_1, \dots, x_m) \longmapsto u_{\alpha_1}(x_1) \cdots u_{\alpha_m}(x_m).$$

This is the root-subgroup factorization theorem; see [Spr]. A proof uses the filtration by root height and the commutator relations. It is an isomorphism of varieties, rather than an assertion that U is an abelian group.

Corollary 1.46. *For $\alpha \in R$, the following are equivalent:*

$$\alpha \in R^+(B), \quad U_\alpha \subset B, \quad \mathfrak{g}_\alpha \subset \mathfrak{b}.$$

If $r = \dim T$, then

$$\dim B = r + \frac{|R|}{2}, \quad \dim G = r + |R|, \quad \dim G/B = \frac{|R|}{2}.$$

Proof. The factorization gives $\text{Lie } U = \bigoplus_{\alpha \in R^+(B)} \mathfrak{g}_\alpha$. Use $B = T \ltimes U$, the root-space decomposition of $\mathfrak{g}$, and $|R^+(B)| = |R|/2$. $\square$

Corollary 1.47. *The derived group G^{der} is semisimple, and multiplication gives a central isogeny*

$$Z(G)^0 \times G^{\text{der}} \longrightarrow G.$$

In particular, $G = R(G)G^{\text{der}}$.

Proof. The structure theorem for reductive groups says that the root subgroups lie in G^{der} and generate a semisimple group whose maximal torus is generated by the coroots. The Lie algebras of this group and $Z(G)^0$ together span $\mathfrak{g}$, so multiplication is surjective. Its kernel is the finite central subgroup $\{(z, z^{-1}) : z \in Z(G)^0 \cap G^{\text{der}}\}$, finite by Proposition 1.40. $\square$

2. REPRESENTATION THEORY OF td -GROUPS

2.1. l -spaces and l -groups.

Definition 2.1. An l -space is a Hausdorff, locally compact space with a basis of compact open sets. An l -group is a Hausdorff topological group with a neighbourhood basis of the identity consisting of compact open subgroups. By van Dantzig's theorem, this is equivalent to being locally compact and totally disconnected.

For an l -space X , write $C_c^\infty(X)$ for the complex-valued, locally constant, compactly supported functions on X . The superscript ∞ means locally constant here. For example, F^* and $\text{GL}_n(F)$ are l -spaces when F is a non-Archimedean local field. All representations below are on complex vector spaces; no topology on the representation space is required.

Definition 2.2. A representation (π, V) of an l -group G is *smooth* if every $v \in V$ is fixed by an open subgroup of G . Equivalently,

$$V = \bigcup_K V^K, \quad V^K = \{v \in V : \pi(k)v = v \text{ for all } k \in K\},$$

where K ranges over compact open subgroups. We denote the category of smooth representations and G -equivariant linear maps by Alg_G .

Lemma 2.3. *For a compact open subgroup K , the functor $V \mapsto V^K$ on Alg_G is exact.*

Proof. Give K Haar measure of total mass 1. The averaging operator

$$p_K v = \int_K \pi(k)v \, dk$$

is a well-defined finite sum on each smooth vector and is a projection onto V^K . If $V_2 \rightarrow V_3$ is a surjection and $v_3 \in V_3^K$, average any lift of v_3 to obtain a K -fixed lift. Left exactness follows directly from the definition. $\square$

Definition 2.4. A smooth representation V is *admissible* if $\dim_{\mathbb{C}} V^K < \infty$ for every compact open subgroup K . Its *smooth contragredient* is

$$\tilde{V} = (V^*)^{\infty}, \quad \langle \tilde{\pi}(g)\ell, v \rangle = \langle \ell, \pi(g^{-1})v \rangle,$$

where $V^* = \text{Hom}_{\mathbb{C}}(V, \mathbb{C})$ and the superscript denotes the smooth vectors for this action.

Proposition 2.5. *For every smooth V and compact open K , restriction gives $(\tilde{V})^K \simeq (V^K)^*$. Consequently, $\tilde{V}$ separates points of V . If V is admissible, so is $\tilde{V}$, and the evaluation map $V \rightarrow \tilde{\tilde{V}}$ is an isomorphism.*

Proof. Extend $\ell \in (V^K)^*$ to V by $v \mapsto \ell(p_K v)$. This is a K -fixed functional, and every K -fixed functional has this form. A nonzero vector belongs to some V^K , so such functionals separate points. Under admissibility the map to the double smooth dual restricts to the usual finite-dimensional bidual isomorphism on each V^K ; taking the union over K proves the last assertion. $\square$

3. THEORY OF ADMISSIBLE REPRESENTATIONS OF p -ADIC GROUPS

Let F be a finite extension of $\mathbb{Q}_p$, with ring of integers $\mathfrak{o}$, maximal ideal $\mathfrak{p} = (\varpi)$ and residue field of cardinality q . Normalize the absolute value by $|\varpi|_F = q^{-1}$. Put

$$G_n = \text{GL}_n(F) = \{g \in M_n(F) : \det g \neq 0\}, \quad T_n = \{\text{diag}(t_1, \dots, t_n) : t_i \in F^{\times}\},$$

$$Z_n = \{aI_n : a \in F^{\times}\}, \quad U_n = \{u \in G_n : u_{ii} = 1, u_{ij} = 0 \text{ for } i > j\}.$$

The upper triangular subgroup is $B_n = T_n \ltimes U_n$. These are closed subgroups of G_n . Here they denote groups of F -points; the corresponding algebraic quotient $\mathbf{G}_n/\mathbf{B}_n$ is the flag variety from Section 1.

Put $K_0 = \text{GL}_n(\mathfrak{o})$ and, for $j \geq 1$, put

$$K_j = \ker(\text{GL}_n(\mathfrak{o}) \longrightarrow \text{GL}_n(\mathfrak{o}/\mathfrak{p}^j)) = I_n + \mathfrak{p}^j M_n(\mathfrak{o}).$$

Lemma 3.1. *The group G_n is locally compact and totally disconnected. The subgroups K_j form a compact open neighbourhood basis at I_n .*

Proof. The determinant realizes G_n as an open subset of the locally compact, totally disconnected space $M_n(F)$. The set K_0 is a closed subset of the compact space $M_n(\mathfrak{o})$, since its determinant belongs to the closed set $\mathfrak{o}^{\times}$; it is also open in G_n . The congruence conditions defining K_j give compact open subgroups and a basis at I_n . $\square$

3.1. Bruhat decomposition. The normalizer $N_{G_n}(T_n) = \{g : gT_ng^{-1} = T_n\}$ consists of monomial matrices, and the centralizer of T_n is T_n . Thus

$$W_n = N_{G_n}(T_n)/T_n \simeq S_n.$$

We represent $w \in S_n$ by its permutation matrix. Gaussian elimination gives the disjoint Bruhat decomposition

$$G_n = \coprod_{w \in S_n} B_n w B_n = \coprod_{w \in S_n} U_n T_n w U_n.$$

The longest element is $w_0 e_i = e_{n+1-i}$, represented by the anti-diagonal matrix with all anti-diagonal entries equal to 1. It indexes the open Bruhat cell; the union runs over every $w \in S_n$.

We shall also use the *Cartan decomposition*, obtained by elementary divisors over $\mathfrak{o}$:

$$G_n = \coprod_{e_1 \geq \dots \geq e_n} K_0 \operatorname{diag}(\varpi^{e_1}, \dots, \varpi^{e_n}) K_0, \quad e_i \in \mathbb{Z}.$$

It follows that K_0 is a maximal compact subgroup. Indeed, if a compact subgroup C contains K_0 , the Cartan decomposition of $g \in C$ places its diagonal factor a in C . Unless $a = I_n$, the set $\{a^r : r \in \mathbb{Z}\}$ has an unbounded matrix entry and cannot be contained in a compact subset of $M_n(F)$.

3.2. Fundamental facts on admissible representations.

3.2.1. The Haar measure. Let G be an l -group and fix a left Haar measure dg . Our modular convention is

$$\int_G f(xg) dx = \Delta_G(g)^{-1} \int_G f(x) dx.$$

The group is *unimodular* if $\Delta_G = 1$. For G_n a Haar measure is a constant multiple of

$$dg = |\det g|_F^{-n} \prod_{i,j} dg_{ij}.$$

Left and right multiplication by $a \in G_n$ both scale additive matrix measure by $|\det a|_F^n$, so this measure is invariant on both sides. In particular, G_n is unimodular. We usually normalize $\operatorname{vol}(K_0) = 1$.

For a closed subgroup $H \subset G$, put $\delta_{H \setminus G} = \Delta_G|_H / \Delta_H$. The appropriate space for quotient integration consists of locally constant functions $F : G \rightarrow \mathbb{C}$ such that

$$F(hg) = \delta_{H \setminus G}(h) F(g), \quad \operatorname{supp}(F) \subset HC_F$$

for some compact $C_F \subset G$. Compact support is required modulo H . There is a right G -invariant integral on these quotient densities. When G is unimodular, compatible Haar measures characterize it by

$$\int_{H \setminus G} \left(\int_H \phi(hg) dh \right) d\dot{g} = \int_G \phi(g) dg, \quad \phi \in C_c^\infty(G).$$

If $\Delta_G|_H = \Delta_H$, these are ordinary functions on $H \setminus G$ and $d\dot{g}$ is an invariant measure. For a general H , the density notation records the required covariance. This will be used for the pairing of induced representations.

3.2.2. The Hecke algebra.

Definition 3.2. The *Hecke algebra* of G is $\mathcal{H}(G) = C_c^\infty(G)$ with convolution

$$(f_1 * f_2)(g) = \int_G f_1(x) f_2(x^{-1}g) dx.$$

For a compact open subgroup K , put $e_K = \operatorname{vol}(K)^{-1} \mathbf{1}_K$ and

$$\mathcal{H}(G//K) = e_K * \mathcal{H}(G) * e_K.$$

This is the algebra of compactly supported, bi- K -invariant functions; its identity is e_K .

For a smooth representation (π, V) , define

$$\pi(f)v = \int_G f(g) \pi(g)v dg.$$

The integral is a finite sum, and $\pi(f_1 * f_2) = \pi(f_1)\pi(f_2)$. Moreover, $\pi(e_K) = p_K$, so e_K projects V onto V^K . Each $f \in \mathcal{H}(G)$ is bi-invariant under some sufficiently small compact open K , and therefore

$$\pi(f) = p_K \pi(f) p_K.$$

If V is admissible, this operator has finite rank. Its trace can consequently be computed on the finite-dimensional space V^K .

Remark 3.3. The algebra $\mathcal{H}(G)$ need not have an identity. The idempotents e_K provide local identities: a finite collection of functions, or of smooth vectors, is fixed by a suitable e_K . Smooth representations are equivalently nondegenerate $\mathcal{H}(G)$ -modules, where nondegenerate means $\mathcal{H}(G)V = V$.

3.2.3. Harish-Chandra distribution characters. Throughout this subsection, G is assumed to be unimodular, as is G_n . Let (π, V) be an admissible smooth representation of G . For $f \in \mathcal{H}(G) = C_c^\infty(G)$, choose a compact open subgroup K such that $f = e_K * f * e_K$. Then

$$\pi(f) = \pi(e_K)\pi(f)\pi(e_K), \quad \text{Im } \pi(f) \subset V^K.$$

Admissibility implies that $\pi(f)$ has finite rank, so its trace is well defined.

Definition 3.4. The *distribution character* of π is the distribution

$$\Theta_\pi : \mathcal{H}(G) \longrightarrow \mathbb{C}, \quad \Theta_\pi(f) = \text{tr } \pi(f).$$

Since G is unimodular, this distribution is invariant under conjugation: if $f^g(x) = f(g^{-1}xg)$, then $\pi(f^g) = \pi(g)\pi(f)\pi(g)^{-1}$ and $\Theta_\pi(f^g) = \Theta_\pi(f)$. In general, $\pi(g)$ itself acts on an infinite-dimensional space and need not have a trace; the test function f is essential in this definition.

Proposition 3.5. *Let $\pi_1, \dots, \pi_n$ be irreducible admissible representations of G . They are pairwise inequivalent if and only if $\Theta_{\pi_1}, \dots, \Theta_{\pi_n}$ are linearly independent.*

If $T : V_2 \longrightarrow V_1$ is an equivalence, then $\pi_1(f) = T\pi_2(f)T^{-1}$ for every $f \in \mathcal{H}(G)$, and hence $\Theta_{\pi_1} = \Theta_{\pi_2}$. This proves one direction. We give two proofs of the other direction.

Murnaghan's Approach

We use the Hecke algebra argument in [Mur, §17]. The required consequence of the Jacobson density theorem is the following.

Lemma 3.6. *Let A be a unital complex algebra and let W be a finite-dimensional simple A -module. Then the action map $A \longrightarrow \text{End}_{\mathbb{C}}(W)$ is surjective.*

Proof. Schur's lemma gives $\text{End}_A(W) = \mathbb{C}$. By the Jacobson density theorem, for a basis $w_1, \dots, w_d$ of W and arbitrary $w'_1, \dots, w'_d \in W$, there is an $a \in A$ with $aw_i = w'_i$ for every i . This realizes every linear endomorphism of W . $\square$

We also recall precisely the relation between irreducible representations and their fixed vectors.

Lemma 3.7. *Fix a compact open subgroup K and put $A_K = e_K * \mathcal{H}(G) * e_K = \mathcal{H}(G//K)$. If V is an irreducible smooth representation with $V^K \neq 0$, then V^K is a simple A_K -module. Two such irreducible representations are equivalent if and only if their K -fixed spaces are isomorphic as A_K -modules.*

Proof. Write $A = \mathcal{H}(G)$ and $e = e_K$. If $0 \neq W \subset eV$ is an eAe -submodule, irreducibility gives $AW = V$. Consequently $eV = eAW = eAeW = W$, proving simplicity.

For the second assertion, a simple eAe -module $W = eV$ determines the A -module $M = Ae \otimes_{eAe} W$. The map

$$M \longrightarrow V, \quad ae \otimes w \longmapsto aw,$$

is surjective, and $eM \simeq W$. Any proper submodule $N \subset M$ has $eN = 0$: otherwise simplicity gives $eN = eM$, and eM generates M , so $N = M$. The sum of all proper submodules still has zero e -part and is therefore proper. Thus M has a unique simple quotient. This proves that W determines V . Here we use the usual equivalence between smooth G -modules and nondegenerate $\mathcal{H}(G)$ -modules. $\square$

Proof of Proposition 3.5. Suppose that $\pi_1, \dots, \pi_n$ are pairwise inequivalent. Choose nonzero $v_i \in V_i$ and a compact open subgroup K fixing every v_i . Then $W_i = V_i^K$ is a nonzero finite-dimensional simple A_K -module, and the W_i are pairwise nonisomorphic by Lemma 3.7.

Let I_i be the kernel of the action of A_K on W_i . By the density lemma,

$$A_K/I_i \simeq \text{End}_{\mathbb{C}}(W_i).$$

Thus the I_i are maximal two-sided ideals. They are distinct: a full matrix algebra has only one simple module up to isomorphism, so $I_i = I_j$ would imply $W_i \simeq W_j$. They are therefore pairwise comaximal, and the Chinese remainder theorem gives a surjection

$$A_K \longrightarrow \prod_{i=1}^n \text{End}_{\mathbb{C}}(W_i), \quad f \longmapsto (\pi_i(f)|_{W_i})_{i=1}^n.$$

For each j , choose $f_j \in A_K$ mapping to the identity in the j -th factor and to zero in every other factor. Since $\pi_i(f_j) = \pi_i(e_K)\pi_i(f_j)\pi_i(e_K)$, its trace on V_i is its trace on W_i . Hence

$$\Theta_{\pi_i}(f_j) = \delta_{ij} \dim W_j.$$

Evaluating a relation $\sum_i a_i \Theta_{\pi_i} = 0$ at f_j gives $a_j \dim W_j = 0$. Thus every a_j is zero. $\square$

Silberger's Approach

The second proof uses matrix coefficients; compare [Sil, §1.11]. For a function f on G , write

$$(\rho(y)f)(x) = f(xy), \quad (\lambda(y)f)(x) = f(y^{-1}x).$$

These are the right and left regular actions. We continue to use the unimodularity of G .

Definition 3.8. Let $(\tilde{\pi}, \tilde{V})$ be the smooth contragredient of a smooth representation (π, V) . For $v \in V$ and $\tilde{v} \in \tilde{V}$, its *matrix coefficient* is

$$f_{\tilde{v},v}(x) = \langle \tilde{v}, \pi(x)v \rangle.$$

We denote the coefficient space by

$$A(\pi) = \text{Span}_{\mathbb{C}}\{f_{\tilde{v},v} : v \in V, \tilde{v} \in \tilde{V}\}.$$

For admissible V , let $\text{End}^0(V)$ consist of the operators T for which some compact open K satisfies

$$\pi(k)T = T = T\pi(k) \quad (k \in K).$$

Such an operator satisfies $T = \pi(e_K)T\pi(e_K)$ and therefore has finite rank. It can be written as a finite sum of operators $v \otimes \tilde{v}$, where $(v \otimes \tilde{v})(w) = \langle \tilde{v}, w \rangle v$. Indeed, choose a basis of its image and note that its coordinate functionals factor through $V \longrightarrow V^K$, so they are smooth. Conversely, every such finite sum belongs to $\text{End}^0(V)$. For $T \in \text{End}^0(V)$, put

$$f_T(x) = \text{tr}(\pi(x)T).$$

For a rank-one operator,

$$f_{v \otimes \tilde{v}}(x) = \langle \tilde{v}, \pi(x)v \rangle.$$

It follows that $T \mapsto f_T$ maps $\text{End}^0(V)$ onto $A(\pi)$.

Lemma 3.9. *For every smooth representation π , $y \in G$, and $f \in \mathcal{H}(G)$,*

$$\pi(\lambda(y)f) = \pi(y)\pi(f), \quad \pi(\rho(y)f) = \pi(f)\pi(y^{-1}).$$

Proof. Both formulas follow by changing variables in the defining integral for $\pi(f)$; the second uses right invariance of Haar measure. $\square$

Lemma 3.10. *If π is irreducible, $A(\pi)$ is a semisimple right G -module all of whose simple constituents are isomorphic to π . Consequently, if $\mathcal{B} \subset A(\pi)$ is a right G -submodule and there is a nonzero G -homomorphism $\mathcal{B} \longrightarrow V_1$ with V_1 irreducible, then $\pi_1 \simeq \pi$.*

Proof. For each nonzero $\tilde{v} \in \tilde{V}$, the map

$$T_{\tilde{v}} : V \longrightarrow A(\pi), \quad v \longmapsto f_{\tilde{v},v},$$

is G -equivariant for the right regular action, because $\rho(g)f_{\tilde{v},v} = f_{\tilde{v},\pi(g)v}$. It is nonzero: choose v with $\langle \tilde{v}, v \rangle \neq 0$ and evaluate at the identity. Irreducibility makes it injective. Thus $A(\pi)$ is a sum of simple submodules isomorphic to V , hence is semisimple and π -isotypic. Every submodule has the same property, so any irreducible quotient of such a submodule is isomorphic to V . $\square$

Remark 3.11. The coefficient space $A(\pi)$ need not be irreducible under right translations: it generally contains several copies of π . The semisimple, isotypic statement is what the proof requires.

Lemma 3.12. *If $\pi_1, \dots, \pi_n$ are pairwise inequivalent irreducible smooth representations, their co-efficient spaces have direct sum:*

$$A(\pi_1) + \dots + A(\pi_n) = A(\pi_1) \oplus \dots \oplus A(\pi_n).$$

Proof. Proceed by induction on n . If $A(\pi_1) \cap \bigoplus_{i=2}^n A(\pi_i)$ were nonzero, it would contain a simple submodule $S \simeq V_1$, by Lemma 3.10. Some coordinate projection of S into $A(\pi_j)$ would be nonzero. Its image would be a simple submodule isomorphic both to V_1 and to V_j , contrary to the inequivalence assumption. $\square$

Second proof of Proposition 3.5. Suppose that $\sum_i a_i \Theta_{\pi_i} = 0$. Fix $f \in \mathcal{H}(G)$ and put $T_i = \pi_i(f) \in \text{End}^0(V_i)$. Lemma 3.9 gives

$$\Theta_{\pi_i}(\lambda(x)f) = \text{tr}(\pi_i(x)T_i) = f_{T_i}(x).$$

Evaluating the assumed distributional relation at $\lambda(x)f$ for every $x \in G$ gives the relation of functions

$$\sum_{i=1}^n a_i f_{T_i} = 0.$$

By Lemma 3.12, $a_i f_{T_i} = 0$ for every i and every f . For a fixed i , choose compact open K with $V_i^K \neq 0$ and take $f = e_K$. Then

$$f_{T_i}(1) = \text{tr } \pi_i(e_K) = \dim V_i^K > 0.$$

Thus $a_i = 0$. Repeating this for every i proves linear independence. $\square$

3.3. The induction process. Let H be a closed subgroup of an l -group G , and let $(\rho, V_\rho) \in \text{Alg}_H$.

Definition 3.13. Let $L(G, \rho)$ be the space of functions $f : G \rightarrow V_\rho$ satisfying

$$f(hg) = \rho(h)f(g) \quad (h \in H, g \in G),$$

and such that $f(gk) = f(g)$ for all $g \in G$ and all k in some compact open subgroup K_f of G . Right translation,

$$(\pi(g_0)f)(g) = f(gg_0),$$

defines the *smooth induced representation* $\text{Ind}_H^G \rho$. The subspace $S(G, \rho)$ of functions whose support is compact modulo H defines the *compactly induced representation* $\text{ind}_H^G \rho$, also written $\text{c-Ind}_H^G \rho$. These are unnormalized inductions.

An H -map $V_\rho \rightarrow V_{\rho'}$ acts on these spaces pointwise. Thus induction and compact induction are functors from Alg_H to Alg_G .

Lemma 3.14. *Let K be compact open in G , let Ω be a set of representatives for $H \backslash G / K$, and put $H_g = H \cap gKg^{-1}$. Evaluation on Ω gives vector-space isomorphisms*

$$L(G, \rho)^K \simeq \prod_{g \in \Omega} V_\rho^{H_g}, \quad S(G, \rho)^K \simeq \bigoplus_{g \in \Omega} V_\rho^{H_g}.$$

Proof. If $h \in H_g$, then $hg = gk$ for some $k \in K$, so $\rho(h)f(g) = f(hg) = f(g)$. Conversely, given $v_g \in V_\rho^{H_g}$, define $f(hgk) = \rho(h)v_g$. The invariance condition makes this well-defined. The images of the double cosets HgK form a disjoint open cover of $H \backslash G$; a compact subset meets only finitely many members. Each image is itself compact, being the image of gK . Consequently compact support modulo H is equivalent to finite support on Ω . $\square$

Proposition 3.15. (a) *Both induction functors are exact over $\mathbb{C}$.*

(b) *If $L \subset H \subset G$ are closed subgroups and $\tau \in \text{Alg}_L$, then*

$$\text{ind}_H^G(\text{Ind}_L^H \tau) \simeq \text{Ind}_L^G \tau, \quad \text{ind}_H^G(\text{ind}_L^H \tau) \simeq \text{ind}_L^G \tau.$$

(c) *With the modular-function convention fixed above,*

$$\widetilde{\text{ind}_H^G \rho} \simeq \text{Ind}_H^G \left(\frac{\Delta_G|_H}{\Delta_H} \otimes \widetilde{\rho} \right).$$

Here the quotient means $h \mapsto \Delta_G(h)/\Delta_H(h)$.

Proof. (a) Taking invariants under a compact subgroup is exact: a lift of an invariant vector can be averaged to obtain an invariant lift. Apply this to each H_g in Lemma 3.14. Products and direct sums of surjective maps of vector spaces are surjective, so the induced maps are surjective on K -fixed vectors. Every smooth vector is fixed by some K , which proves the assertion.

(b) The maps are evaluation and currying:

$$F \mapsto (g \mapsto F(g)(1)), \quad f \mapsto (g \mapsto (h \mapsto f(hg))).$$

They preserve covariance and right smoothness. For compact induction, compactness of the support modulo L implies compactness modulo H . Conversely, a right K -invariant F of compact support modulo H is determined by finitely many values $F(g)$, each compactly supported modulo L in H . These finitely many supports give compact support modulo L in G .

(c) Put $\delta_{H \backslash G}(h) = \Delta_G(h) / \Delta_H(h)$. If $f \in S(G, \rho)$ and $\tilde{f} \in L(G, \delta_{H \backslash G} \tilde{\rho})$, then

$$\varphi(g) = \langle \tilde{f}(g), f(g) \rangle \quad \text{satisfies} \quad \varphi(hg) = \delta_{H \backslash G}(h) \varphi(g).$$

It is a compactly supported density on $H \backslash G$. Integrating this density gives a G -invariant pairing

$$\langle \tilde{f}, f \rangle = I_{H \backslash G}(\varphi).$$

At the level of K -fixed vectors, this is the pairing of the direct sum in Lemma 3.14 with the product of the dual spaces. Indeed, $\widehat{V}_\rho^{H_g} = (V_\rho^{H_g})^*$ by averaging, and the modular characters are trivial on H_g . The weights of the individual double cosets in the pairing are nonzero. Thus this pairing identifies the K -fixed vectors on the left with the full dual of $S(G, \rho)^K$, for every K . Taking the union over K gives the claimed smooth dual. See also [Mur, Section 5]. $\square$

Remark 3.16. The notation $I_{H \backslash G}$ does not assert that there is an invariant measure on the ordinary quotient. Such a measure exists when $\Delta_G|_H = \Delta_H$; in general it is the density, with the covariance displayed above, that can be integrated invariantly. If $H \backslash G$ is compact, then $\text{ind}_H^G \rho = \text{Ind}_H^G \rho$.

Proposition 3.17 (Frobenius reciprocity). *For smooth π and ρ , evaluation at 1 induces*

$$\text{Hom}_G(\pi, \text{Ind}_H^G \rho) \simeq \text{Hom}_H(\pi|_H, \rho).$$

Proof. The inverse sends an H -map T to $v \mapsto (g \mapsto T(\pi(g)v))$. If v is fixed by K , the resulting function is right K -invariant. The two constructions are inverse. $\square$

3.3.1. Parabolic induction and Jacquet modules. Return to $G = G_n = \text{GL}_n(F)$. An *ordered composition* $\underline{n} = (n_1, \dots, n_r)$ of n has $n_i > 0$ and $\sum_i n_i = n$. Let $P_{\underline{n}}$ be the subgroup of block upper triangular matrices with diagonal block sizes n_i . It has a Levi decomposition

$$P_{\underline{n}} = M_{\underline{n}} U_{\underline{n}}, \quad M_{\underline{n}} \simeq \prod_{i=1}^r \text{GL}_{n_i}(F),$$

where $U_{\underline{n}}$ consists of block upper triangular matrices with identity diagonal blocks. Every parabolic subgroup is conjugate to a standard one of this form.

Definition 3.18. Let U be the group of F -points of a unipotent algebraic subgroup and $\theta : U \rightarrow \mathbb{C}^\times$ a smooth character. Set

$$V(U, \theta) = \text{Span}_{\mathbb{C}} \{ \pi(u)v - \theta(u)v : u \in U, v \in V \}.$$

If M normalizes U , put

$$M_\theta = \{ m \in M : \theta(mum^{-1}) = \theta(u) \text{ for every } u \in U \}.$$

The *twisted Jacquet module* is $J_{U, \theta}(V) = V / V(U, \theta)$, with the action $m[v] = [\pi(m)v]$ of M_θ . For $\theta = 1$ we write V_U and π_U .

The action is well-defined because

$$\pi(m)(\pi(u)v - \theta(u)v) = \pi(mum^{-1})\pi(m)v - \theta(mum^{-1})\pi(m)v.$$

For $P = MU$ a parabolic subgroup, $M_1 = M$, so V_U is a smooth M -representation.

Proposition 3.19. *The functor $V \mapsto J_{U,\theta}(V)$ is exact over $\mathbb{C}$. Moreover, $[v] = 0$ in this quotient if and only if*

$$e_{C,\theta}v := \frac{1}{\text{vol}(C)} \int_C \theta(u)^{-1} \pi(u)v \, du = 0$$

for some compact open subgroup C of U .

Proof. A unipotent F -group is the union of an increasing sequence of compact open subgroups. On each such subgroup C , twisted averaging identifies coinvariants with the θ -isotypic subspace and is exact, by the same averaging argument used for invariants. The twisted coinvariant space for U is the filtered direct limit of these spaces. Filtered direct limits of vector spaces are exact. The last assertion is the criterion for a vector to vanish in this direct limit. $\square$

For $P = MU$ define

$$\delta_P(m) = |\det(\text{Ad}(m)|_{\mathfrak{u}})|_F = \Delta_P(m)^{-1}.$$

For $m = \text{diag}(m_1, \dots, m_r) \in M_{\underline{n}}$,

$$\delta_{P_{\underline{n}}}(m) = \prod_{i < j} |\det m_i|_F^{n_j} |\det m_j|_F^{-n_i}.$$

Extend a representation σ of M trivially across U . The *normalized parabolic induction* and *normalized Jacquet functor* are

$$i_P^G \sigma = \text{Ind}_P^G(\delta_P^{1/2} \sigma), \quad r_P^G \pi = \delta_P^{-1/2} \pi_U.$$

Since $P \backslash G$ is compact, both versions of parabolic induction have compact support modulo P . The duality formula gives

$$\widetilde{i_P^G \sigma} \simeq i_P^G \widetilde{\sigma}, \quad \text{Hom}_G(\pi, i_P^G \sigma) \simeq \text{Hom}_M(r_P^G \pi, \sigma).$$

The second identity follows from Frobenius reciprocity because any P -map to a representation trivial on U factors through V_U .

If σ is admissible, then $i_P^G \sigma$ is admissible. Indeed, $P \backslash G/K$ is finite, and Lemma 3.14 expresses its K -fixed space as a finite direct sum of invariant spaces for compact open subgroups of P . Their images in $M = P/U$ are compact open, so these spaces are finite dimensional by admissibility of σ .

3.4. Supercuspidal representations. Write $Z = Z(G)$ and $G_0 = \{g \in G : |\det g|_F = 1\}$. This is an open normal subgroup, and its center is the compact group $Z \cap G_0 = \mathfrak{o}^\times I_n$.

Definition 3.20. A smooth representation (π, V) is *supercuspidal* if every matrix coefficient $g \mapsto \langle \tilde{v}, \pi(g)v \rangle$ has support contained in ZC for some compact subset C of G . This definition is allowed for reducible smooth representations here.

Remark 3.21. A locally constant function of compact support takes finitely many values. Compact support modulo Z does not give this conclusion on G . Its restriction to G_0 does have compact support: the possible absolute values of a scalar z with $zc \in G_0$, $c \in C$, form a finite set, and the corresponding scalars lie in a compact subset of $F^\times$.

Lemma 3.22. *If π is supercuspidal and $\pi|_{G_0}$ is finitely generated, then π is admissible.*

Proof. Let K be compact open in G_0 and fix $v \in V$. If

$$S_v = \{\pi(e_K)\pi(g)v : g \in G_0\}$$

spanned an infinite dimensional space, we could choose linearly independent $w_j = \pi(e_K)\pi(g_j)v$, $j \geq 1$. Choose a linear functional ℓ on V^K with $\ell(w_j) = j$. Then $\tilde{v} = \ell \circ \pi(e_K)$ belongs to $\tilde{V}^K$, but its matrix coefficient at g_j is j . This contradicts compact support on G_0 and local constancy. Thus $\text{Span } S_v$ is finite dimensional. Applying this to finitely many G_0 -generators gives $\dim V^K < \infty$. Every compact subgroup of G lies in G_0 , since its determinant valuation is a compact subgroup of $\mathbb{Z}$ and hence is zero. This proves admissibility for G as well. $\square$

Lemma 3.23. *The following conditions are equivalent:*

- (1) π is supercuspidal;
- (2) $\pi|_{G_0}$ is supercuspidal;
- (3) for every compact open $K \subset G$ and every $v \in V$, the function $g \mapsto \pi(e_K)\pi(g)v$ has compact support modulo Z .

Proof. The preceding remark proves (1) $\Rightarrow$ (2). The smooth dual for G_0 is the same vector space as the smooth dual for G , since G_0 is open.

For (2) $\Rightarrow$ (3), first work on G_0 with $W = \text{Span}(G_0v)$. Every smooth matrix coefficient of W extends to one of $V|_{G_0}$: extend a functional on W^K to V^K and compose with $\pi(e_K)$. The averaging argument in Lemma 3.22, applied to this cyclic representation of G_0 , shows that W^K is finite dimensional. Choose a basis of $(W^K)^*$. The coordinates of $\pi(e_K)\pi(g)v$, $g \in G_0$, are finitely many compactly supported scalar matrix coefficients. Their supports have compact union.

The quotient G/ZG_0 is finite: the determinant valuation identifies it with $\mathbb{Z}/n\mathbb{Z}$. Choose representatives $a_1, \dots, a_n$. For $z \in Z$ and $h \in G_0$,

$$\pi(e_K)\pi(za_ih)v = \pi(za_i)\pi(e_{a_i^{-1}Ka_i})\pi(h)v.$$

The right side vanishes outside finitely many compact subsets of G_0 , uniformly in z . This proves (3). Finally, if $\tilde{v}$ is fixed by K , then

$$\langle \tilde{v}, \pi(g)v \rangle = \langle \tilde{v}, \pi(e_K)\pi(g)v \rangle.$$

Thus (3) implies (1). □

Theorem 3.24 (Harish-Chandra). *A smooth representation π of $\text{GL}_n(F)$ is supercuspidal if and only if $V_U = 0$ for the unipotent radical U of every proper parabolic subgroup of G .*

Proof. It is enough to use maximal proper parabolics. If $P_1 \subset P$, then their unipotent radicals satisfy $U_1 \supset U$, so $V_U = 0$ implies $V_{U_1} = 0$. Every maximal parabolic is conjugate to $P_{\ell, n-\ell}$, $1 \leq \ell < n$.

Suppose these Jacquet modules vanish. Use the Cartan decomposition

$$G_0 = \bigcup_{e \in E} K_0 a_e K_0, \quad a_e = \text{diag}(\varpi^{e_1}, \dots, \varpi^{e_n}),$$

where $E = \{e \in \mathbb{Z}^n : e_1 \geq \dots \geq e_n, \sum_i e_i = 0\}$. Fix v and $\tilde{v}$, and choose a compact open subgroup of K_0 fixing both. Because it has finite index in K_0 , only finitely many translates of v and $\tilde{v}$ occur when evaluating the coefficient on the displayed double cosets. It therefore suffices to bound its support on the a_e for each such pair of vectors.

For $U = U_{\ell, n-\ell}$, the averaging criterion for coinvariants gives a compact open $C_\ell \subset U$ with $e_{C_\ell}v = 0$. For $u = \begin{pmatrix} I_\ell & X \\ 0 & I_{n-\ell} \end{pmatrix}$,

$$a_e u a_e^{-1} = \begin{pmatrix} I_\ell & (\varpi^{e_i - e_j} X_{ij}) \\ 0 & I_{n-\ell} \end{pmatrix}, \quad i \leq \ell < j.$$

Since $e_i - e_j \geq e_\ell - e_{\ell+1}$, there is a bound b_ℓ such that $e_\ell - e_{\ell+1} \geq b_\ell$ implies $a_e C_\ell a_e^{-1} \subset \text{Stab}_U(\tilde{v})$. Consequently

$$\langle \tilde{v}, \pi(a_e)v \rangle = \langle \tilde{v}, e_{a_e C_\ell a_e^{-1}} \pi(a_e)v \rangle = \langle \tilde{v}, \pi(a_e) e_{C_\ell} v \rangle = 0.$$

Only finitely many $e \in E$ have all adjacent differences bounded. The coefficient is therefore compactly supported on G_0 , so Lemma 3.23 proves supercuspidality.

Conversely, suppose π is supercuspidal and fix $U = U_{\ell, n-\ell}$ and $v \in V$. Choose a sufficiently small principal congruence subgroup K fixing v . Block Gaussian elimination gives its Iwahori decomposition

$$K = K^+ K^M K^-, \quad K^+ = K \cap U, \quad K^M = K \cap M, \quad K^- = K \cap U^-,$$

and the corresponding averaging factorization $e_K = e_{K^+} e_{K^M} e_{K^-}$. Put

$$a = \text{diag}(\varpi^{n-\ell} I_\ell, \varpi^{-\ell} I_{n-\ell}) \in G_0.$$

The sequence a^m escapes compact subsets modulo Z . Lemma 3.23 gives $\pi(e_K)\pi(a^m)v = 0$ for all sufficiently large m , and hence $e_{a^{-m}Ka^m}v = 0$. Conjugation by a^{-m} contracts U^- , while a centralizes M . For large m the last two factors in the conjugated averaging factorization fix v , so

$$0 = e_{a^{-m}Ka^m}v = e_{a^{-m}K+a^m}v.$$

The averaging criterion gives $v \in V(U)$. Since v was arbitrary, $V_U = 0$. $\square$

Corollary 3.25. *An irreducible smooth representation is supercuspidal if and only if it is not a subquotient of $i_P^G \sigma$ for any proper parabolic $P = MU$ and any irreducible smooth representation σ of M .*

Proof. This is the cuspidal-support theorem, together with the exactness of Jacquet functors and the geometric lemma; see [BZ, Cas]. For the easier embedding criterion, Frobenius reciprocity already shows that a supercuspidal representation cannot embed in such an induction. The assertion about arbitrary subquotients uses the cited theorem. $\square$

4. BERNSTEIN–ZELEVINSKY

4.1. Zelevinsky derivatives.

4.1.1. *The mirabolic subgroup and normalization.* Fix a nontrivial smooth additive character $\psi : F \rightarrow \mathbb{C}^\times$. Put $G_m = \mathrm{GL}_m(F)$ for $m \geq 1$, and use $\mathrm{GL}_0(F) = \{1\}$ when a derivative has rank zero. For $n \geq 2$, the *mirabolic subgroup* is

$$\mathcal{P}_n = \left\{ \begin{pmatrix} g & x \\ 0 & 1 \end{pmatrix} : g \in G_{n-1}, x \in F^{n-1} \right\} = G_{n-1} \ltimes V_n, \quad V_n = \left\{ u(x) = \begin{pmatrix} I_{n-1} & x \\ 0 & 1 \end{pmatrix} \right\}.$$

Here G_{n-1} is embedded by $g \mapsto \mathrm{diag}(g, 1)$. The mirabolic subgroup is not a parabolic subgroup of G_n . Set $\mathcal{P}_1 = \{1\}$ and write $\nu_m(g) = |\det g|_F$ on G_m or its subgroups.

Define a character of V_n by

$$\theta_n(u(x)) = \psi(x_{n-1}).$$

Since $gu(x)g^{-1} = u(gx)$, its stabilizer in G_{n-1} is

$$\{g \in G_{n-1} : e_{n-1}g = e_{n-1}\} = \mathcal{P}_{n-1}, \quad e_{n-1} = (0, \dots, 0, 1).$$

Thus $Q_n = \mathcal{P}_{n-1}V_n$ is a closed subgroup of $\mathcal{P}_n$, and θ_n extends to it by being trivial on $\mathcal{P}_{n-1}$.

For any smooth $\mathcal{P}_n$ -representation (τ, E) , write

$$E_{V_n, \theta} = E / \mathrm{Span}\{\tau(v)e - \theta(v)e : v \in V_n, e \in E\}.$$

The quotient for $\theta = 1$ is a G_{n-1} -module; the quotient for $\theta = \theta_n$ is a $\mathcal{P}_{n-1}$ -module.

Remark 4.1. The functors below use the *normalized* convention of [BZ, Sections 1.8 and 3]. Conjugation by $g \in G_{n-1}$ on $V_n \simeq F^{n-1}$ scales Haar measure by $\nu_{n-1}(g)$. This is the source of every factor $\nu_{n-1}^{\pm 1/2}$ below. Dropping those factors gives unnormalized functors and changes the resulting G_{n-k} -actions.

4.1.2. *The Bernstein–Zelevinsky functors.* Ranks are included in the notation in this subsection. They will subsequently be suppressed when the source determines them. The following formulas apply for $n \geq 2$; rank one is specified below.

Definition 4.2. The normalized quotient functors are

$$\Psi_n^- : \mathrm{Alg}_{\mathcal{P}_n} \longrightarrow \mathrm{Alg}_{G_{n-1}}, \quad \Phi_n^- : \mathrm{Alg}_{\mathcal{P}_n} \longrightarrow \mathrm{Alg}_{\mathcal{P}_{n-1}},$$

with underlying spaces $E_{V_n, 1}$ and E_{V_n, θ_n} , respectively. Their actions are

$$(\Psi_n^- \tau)(g)[e] = \nu_{n-1}(g)^{-1/2}[\tau(g)e], \quad (\Phi_n^- \tau)(p)[e] = \nu_{n-1}(p)^{-1/2}[\tau(p)e].$$

Definition 4.3. For $\rho \in \text{Alg}_{G_{n-1}}$, define $\Psi_n^+ \rho \in \text{Alg}_{\mathcal{P}_n}$ on the same vector space by

$$(\Psi_n^+ \rho) \left(\begin{pmatrix} g & x \\ 0 & 1 \end{pmatrix} \right) = \nu_{n-1}(g)^{1/2} \rho(g).$$

Thus V_n acts trivially. For $\sigma \in \text{Alg}_{\mathcal{P}_{n-1}}$, let

$$\Phi_n^+ \sigma = \text{ind}_{Q_n}^{\mathcal{P}_n} (\nu_{n-1}^{1/2} \sigma \otimes \theta_n).$$

The inducing representation sends $up \in Q_n$, with $u \in V_n$ and $p \in \mathcal{P}_{n-1}$, to $\theta_n(u) \nu_{n-1}(p)^{1/2} \sigma(p)$.

Concretely, $\Phi_n^+ \sigma$ consists of smooth functions $f : \mathcal{P}_n \rightarrow E_\sigma$ with compact support modulo Q_n and covariance

$$f(upq) = \theta_n(u) \nu_{n-1}(p)^{1/2} \sigma(p) f(q).$$

The $\mathcal{P}_n$ -action is right translation. The support condition is essential: replacing compact induction by full smooth induction gives a different functor,

$$\widehat{\Phi}_n^+ \sigma = \text{Ind}_{Q_n}^{\mathcal{P}_n} (\nu_{n-1}^{1/2} \sigma \otimes \theta_n).$$

At rank one, Ψ_1^- and Ψ_1^+ are the identity functors between representations of $\mathcal{P}_1 = \{1\}$ and $\text{GL}_0(F) = \{1\}$. No $\Phi_1^\pm$ is needed.

Proposition 4.4 (Adjunctions). *All five functors $\Psi_n^\pm$, $\Phi_n^\pm$, and $\widehat{\Phi}_n^+$ are exact. Their adjunctions are*

$$\Psi_n^- \dashv \Psi_n^+, \quad \Phi_n^+ \dashv \Phi_n^- \dashv \widehat{\Phi}_n^+.$$

More explicitly, there are natural isomorphisms

$$\begin{aligned} \text{Hom}_{G_{n-1}}(\Psi_n^- \tau, \rho) &\simeq \text{Hom}_{\mathcal{P}_n}(\tau, \Psi_n^+ \rho), \\ \text{Hom}_{\mathcal{P}_n}(\Phi_n^+ \sigma, \tau) &\simeq \text{Hom}_{\mathcal{P}_{n-1}}(\sigma, \Phi_n^- \tau), \\ \text{Hom}_{\mathcal{P}_{n-1}}(\Phi_n^- \tau, \sigma) &\simeq \text{Hom}_{\mathcal{P}_n}(\tau, \widehat{\Phi}_n^+ \sigma). \end{aligned}$$

The first isomorphism is the universal property of coinvariants; the two half-modulus twists cancel. The third follows from smooth Frobenius reciprocity and the same universal property. The second uses the fact that the nontrivial characters of V_n form a single open orbit. This is the part of smooth imprimitivity used in the following proof.

Theorem 4.5 (The two-orbit decomposition). *There are natural isomorphisms*

$$\Psi_n^- \Psi_n^+ \simeq \text{Id}, \quad \Phi_n^- \Phi_n^+ \simeq \text{Id}, \quad \Phi_n^- \widehat{\Phi}_n^+ \simeq \text{Id},$$

and vanishing identities

$$\Phi_n^- \Psi_n^+ = 0, \quad \Psi_n^- \Phi_n^+ = 0.$$

For every smooth $\mathcal{P}_n$ -representation τ , the adjunction maps give a natural exact sequence

$$0 \longrightarrow \Phi_n^+ \Phi_n^- \tau \longrightarrow \tau \longrightarrow \Psi_n^+ \Psi_n^- \tau \longrightarrow 0.$$

Proof. The smooth character group of V_n is identified with row vectors $\xi \in F^{n-1}$ by $\xi \mapsto (u(x) \mapsto \psi(\xi x))$. The G_{n-1} -action has exactly two orbits: $\{0\}$ and $F^{n-1} \setminus \{0\}$. The stabilizer of e_{n-1} is $\mathcal{P}_{n-1}$.

Fourier transform identifies the convolution algebra $C_c^\infty(V_n)$ with the algebra $C_c^\infty(F^{n-1})$ under pointwise multiplication. A smooth V_n -module is a nondegenerate module over this algebra. Its fiber at ξ is its quotient by $\tau(u(x)) - \psi(\xi x)$; in particular, the fiber at 0 is $E_{V_n,1}$. The quotient map to this fiber has kernel given by the part supported on the open orbit. This assertion is algebraic: compactly supported locally constant functions that vanish at 0 vanish on a neighborhood of 0, and the corresponding idempotents exhaust that kernel.

On the open orbit, equivariance identifies the module with compactly supported sections of the homogeneous bundle determined by its fiber at e_{n-1} . Indeed, the orbit map has local sections, and a compactly supported section can be expressed using finitely many local sections and a compact open partition. With the normalized action on that fiber, this section space is $\Phi_n^+(\Phi_n^- \tau)$. Extension

by zero embeds it in E . The closed-orbit quotient is $\Psi_n^+(\Psi_n^-\tau)$. This gives the displayed exact sequence.

Taking the fiber of an induced open-orbit section space recovers its original fiber, while taking its fiber at 0 gives zero. A module supported at 0 has zero fiber at e_{n-1} . These observations give the composition and vanishing identities. The same fiber calculation applies to full induction, which uses unrestricted smooth sections on the open orbit, giving $\Phi_n^-\widehat{\Phi}_n^+ \simeq \text{Id}$. The corresponding equivalence between open-orbit modules and $\mathcal{P}_{n-1}$ -modules gives the middle adjunction in Proposition 4.4. This is the smooth imprimitivity argument underlying [BZ, Proposition 3.2].

Finally, exactness of Ψ_n^- and Φ_n^- follows from twisted averaging over compact subgroups of V_n , as for Jacquet modules. Inflation and twisting are exact, as are smooth induction and compact induction over $\mathbb{C}$. This proves the remaining exactness assertions. $\square$

Remark 4.6. The last exact sequence need not split. In particular, $\Phi_n^+\Phi_n^-\tau$ is naturally a subrepresentation of τ , whereas $\Psi_n^+\Psi_n^-\tau$ is naturally a quotient. The adjunction $\Phi_n^+ \dashv \Phi_n^-$ concerns this particular character-orbit situation; it is not a general assertion that compact induction from an arbitrary closed subgroup is left adjoint to restriction.

Corollary 4.7. *The functor Φ_n^+ is an equivalence from $\text{Alg}_{\mathcal{P}_{n-1}}$ to the full subcategory of $\text{Alg}_{\mathcal{P}_n}$ consisting of τ with $\Psi_n^-\tau = 0$. Its inverse is Φ_n^- . The functor Ψ_n^+ is an equivalence onto the full subcategory where V_n acts trivially. Both Φ_n^+ and Ψ_n^+ preserve irreducibility.*

Proof. The two equivalences follow from the identities and exact sequence in Theorem 4.5. The open-orbit subcategory is closed under subobjects because Ψ_n^- is exact; the trivial-action subcategory is also closed under subobjects. Consequently a subobject of the image of an irreducible object corresponds to a subobject of that original object. $\square$

4.1.3. The mirabolic filtration.

Definition 4.8. For $\tau \in \text{Alg}_{\mathcal{P}_n}$, its normalized k th derivative is

$$\tau^{(k)} = \Psi_{n-k+1}^- \Phi_{n-k+2}^- \cdots \Phi_n^- \tau \in \text{Alg}_{\text{GL}_{n-k}(F)}, \quad 1 \leq k \leq n.$$

When $k = 1$ the string of Φ^- functors is empty. Equivalently, with changing ranks understood, $\tau^{(k)} = \Psi^-(\Phi^-)^{k-1}\tau$.

Define functors $\mathcal{L}_r : \text{Alg}_{\mathcal{P}_{n-r}} \rightarrow \text{Alg}_{\mathcal{P}_n}$ by

$$\mathcal{L}_0 = \text{Id}, \quad \mathcal{L}_r = \Phi_n^+ \Phi_{n-1}^+ \cdots \Phi_{n-r+1}^+ \quad (1 \leq r \leq n-1).$$

They raise the mirabolic rank by r .

Theorem 4.9 (Mirabolic filtration). *Every smooth $\mathcal{P}_n$ -representation τ has a natural decreasing filtration*

$$\tau = \mathcal{F}^0\tau \supset \mathcal{F}^1\tau \supset \cdots \supset \mathcal{F}^{n-1}\tau \supset \mathcal{F}^n\tau = 0$$

with

$$\mathcal{F}^r\tau = \mathcal{L}_r(\Phi_{n-r+1}^- \cdots \Phi_n^-\tau) \quad (1 \leq r \leq n-1),$$

where these spaces are embedded in τ by the successive adjunction maps. Its quotients are

$$\mathcal{F}^{k-1}\tau / \mathcal{F}^k\tau \simeq \mathcal{L}_{k-1}\Psi_{n-k+1}^+(\tau^{(k)}), \quad 1 \leq k \leq n.$$

Thus derivatives describe the successive quotients of a mirabolic representation, together with the indicated rank-raising functors.

Proof. Theorem 4.5 first gives $\mathcal{F}^1\tau = \Phi_n^+\Phi_n^-\tau$ with quotient $\Psi_n^+\tau^{(1)}$. Apply the same exact sequence to $\Phi_n^-\tau$ and then apply the exact functor Φ_n^+ . Iteration gives the asserted filtration and all the quotient formulas. The final quotient occurs on $\mathcal{P}_1 = \{1\}$, where $\Psi_1^+\Psi_1^-$ is the identity, so the filtration ends at zero without using a rank-one Φ functor. $\square$

Corollary 4.10 (Irreducible mirabolic representations). *For every irreducible smooth representation τ of $\mathcal{P}_n$, there is a unique $k \in \{1, \dots, n\}$ and a unique isomorphism class of irreducible smooth representation ρ of $\mathrm{GL}_{n-k}(F)$ such that*

$$\tau \simeq \mathcal{L}_{k-1} \Psi_{n-k+1}^+ \rho.$$

Its only nonzero derivative is $\tau^{(k)} \simeq \rho$. Conversely every representation of this displayed form is irreducible when ρ is irreducible.

Proof. If $\Psi_n^- \tau \neq 0$, the quotient map in Theorem 4.5 is an isomorphism by irreducibility. Thus $\tau = \Psi_n^+ \rho$ with ρ irreducible. Otherwise $\tau = \Phi_n^+ \Phi_n^- \tau$, and the equivalence in Corollary 4.7 makes $\Phi_n^- \tau$ irreducible. Repeat at the next rank. The procedure must stop by rank one and gives the displayed form. The composition identities show directly that its derivatives vanish except in order k , where the result is ρ . This proves uniqueness. The converse follows because each rank-raising functor preserves irreducibility. $\square$

4.1.4. Derivatives of representations of the general linear group.

Definition 4.11. For a smooth representation π of G_n , put

$$\pi^{(0)} = \pi, \quad \pi^{(k)} = \Psi^-(\Phi^-)^{k-1}(\pi|_{\mathcal{P}_n}), \quad 1 \leq k \leq n.$$

Thus $\pi^{(k)}$ is a representation of $\mathrm{GL}_{n-k}(F)$; for $k = n$ it is a vector space. We call it the normalized *right derivative*. The word right refers to the use of the last columns in the chosen upper triangular convention. Set $\pi^{(k)} = 0$ for $k > n$.

Each fixed-order derivative is exact, since restriction and the quotient functors are exact. Applied to $\pi|_{\mathcal{P}_n}$, the filtration in Theorem 4.9 reconstructs its successive mirabolic quotients from the representations $\pi^{(k)}$.

There is a useful description without intermediate mirabolic groups. For $1 \leq k \leq n$, put $m = n - k$ and let

$$N_{m,k} = \left\{ \begin{pmatrix} I_m & X \\ 0 & u \end{pmatrix} : X \in M_{m \times k}(F), u \in U_k \right\}, \quad \eta_{m,k} \left(\begin{pmatrix} I_m & X \\ 0 & u \end{pmatrix} \right) = \psi \left(\sum_{i=1}^{k-1} u_{i,i+1} \right).$$

An empty sum is zero, so $\eta_{m,1} = 1$. For $m = 0$, these mean $N_{0,n} = U_n$ and its standard nondegenerate character ψ_n .

Proposition 4.12. *The underlying space of $\pi^{(k)}$ is $V_{N_{m,k}, \eta_{m,k}}$. Its $\mathrm{GL}_m(F)$ -action is*

$$g[v] = \nu_m(g)^{-k/2} [\pi(\mathrm{diag}(g, I_k))v].$$

The factor is 1 when $m = 0$.

Proof. The last-column groups occurring successively in the definition generate $N_{m,k}$. The first $k - 1$ quotient steps impose the character on the simple root groups in the bottom-right $k \times k$ block; the final Ψ^- step imposes the trivial character on the remaining rectangle. Taking these successive coinvariants is therefore the displayed single twisted quotient. At every step, restriction to the surviving $\mathrm{GL}_m(F)$ gives one factor $\nu_m^{-1/2}$. There are k steps, giving $\nu_m^{-k/2}$. $\square$

In particular, if $D_{\mathrm{un}}^{(k)} \pi$ denotes the unnormalized derivative, the precise conversion is

$$\pi^{(k)} \simeq \nu_{n-k}^{-k/2} D_{\mathrm{un}}^{(k)} \pi, \quad 1 \leq k < n.$$

The top derivatives have the same underlying vector space under either convention.

4.1.5. *Jacquet modules and Whittaker functionals.* Let $Q_{m,k} = P_{(m,k)}$ be the standard parabolic of G_n with Levi $G_m \times G_k$, and let $R_{m,k}$ be its unipotent radical. This notation is used for $m, k > 0$; when $m = 0$, take $Q_{0,n} = G_n$ and let its Jacquet functor be the identity. Write

$$\mathrm{Wh}_k(W) = W_{U_k, \psi_k}, \quad \psi_k(u) = \psi \left(\sum_{i=1}^{k-1} u_{i,i+1} \right).$$

For a $G_m \times G_k$ -module, this quotient retains the action of the first factor.

Proposition 4.13. *There is a natural isomorphism of $\mathrm{GL}_m(F)$ -modules*

$$\pi^{(k)} \simeq (\mathrm{Id} \otimes \mathrm{Wh}_k) r_{Q_{m,k}}^{G_n}(\pi).$$

Here the right side means taking the Whittaker quotient in the second Levi factor, rather than a tensor product decomposition of the whole Jacquet module.

Proof. One has $N_{m,k} = R_{m,k} \rtimes U_k$, with the indicated embeddings. Thus its twisted quotient is obtained by first taking ordinary $R_{m,k}$ -coinvariants and then (U_k, ψ_k) -coinvariants. For $g \in G_m$ and $h \in G_k$,

$$\delta_{Q_{m,k}}(\mathrm{diag}(g, h)) = \nu_m(g)^k \nu_k(h)^{-m}.$$

The normalized Jacquet action on the first factor is therefore $\nu_m(g)^{-k/2}$ times the inherited action. This is exactly the factor in Proposition 4.12. $\square$

This description is also given in [CS, Section 4.2]. It explains why a derivative retains one factor of a Jacquet module: the other factor is replaced by its Whittaker quotient.

Proposition 4.14. *The top derivative and its full algebraic dual satisfy*

$$\pi^{(n)} = V_{U_n, \psi_n}, \quad (\pi^{(n)})^* \simeq \mathrm{Hom}_{U_n}(\pi, \psi_n).$$

For an irreducible smooth representation, the dimension of this space is at most one. Such a representation is called *generic* if the space is nonzero.

Proof. The first identity is Proposition 4.12 with $m = 0$. A linear functional on the quotient is exactly a functional ℓ on V satisfying $\ell(\pi(u)v) = \psi_n(u)\ell(v)$. This proves the dual identification. The dimension bound is the uniqueness theorem for Whittaker functionals on irreducible representations of G_n ; see [BZ, Section 4] and [Zel, Section 8]. $\square$

A nonzero Whittaker functional gives functions

$$W_v(g) = \ell(\pi(g)v), \quad W_v(ug) = \psi_n(u)W_v(g).$$

The map $v \mapsto W_v$ is G_n -equivariant for right translation and is injective when π is irreducible. Its image is the Whittaker model. Thus $\pi^{(n)} \neq 0$ is the existence criterion for this model, and uniqueness says that its defining functional is unique up to a scalar.

Proposition 4.15. *If π is a smooth representation of finite length, every derivative $\pi^{(k)}$ has finite length and is admissible as a representation of $\mathrm{GL}_{n-k}(F)$. Its restriction $\pi|_{\mathcal{P}_n}$ also has finite length.*

Proof. For $0 < k < n$, the normalized Jacquet module has finite length. Its irreducible factors are external tensor products $\sigma \boxtimes \tau$ of irreducible admissible representations of G_{n-k} and G_k . Taking Whittaker coinvariants in the second factor is exact and sends such a factor to σ or 0, by Whittaker uniqueness. Apply Proposition 4.13 to a composition series. The top derivative is finite dimensional by the same argument applied to a composition series of π . Finite-length smooth representations of general linear groups are admissible, giving the assertion about admissibility. Finally the mirabolic filtration has finitely many layers, and the rank-raising functors preserve finite length by Corollary 4.7. See also [BZ, Sections 3–4]. $\square$

Remark 4.16. Admissibility in this proposition concerns the smaller general linear groups. Restriction to $\mathcal{P}_n$ need not be admissible. Also, admissibility alone does not imply that the top derivative is finite dimensional: for $G_1 = F^\times$, the direct sum of characters whose conductors tend to infinity is admissible but has an infinite dimensional first derivative.

Definition 4.17. For $\pi \neq 0$ on G_n with $n \geq 1$, let $d(\pi) = \max\{k : \pi^{(k)} \neq 0\}$. The representation $\pi^{(d(\pi))}$ is its *highest derivative*. The mirabolic filtration guarantees that some positive derivative is nonzero.

Theorem 4.18 (Zelevinsky). *The highest derivative of an irreducible smooth representation of G_n is irreducible.*

This is [Zel, Theorem 8.1]. Intermediate derivatives can have several irreducible constituents. Generic irreducible representations have $d(\pi) = n$ and top derivative $\mathbb{C}$.

4.1.6. *The Leibniz rule.* For representations of G_a and G_b , use the notation

$$\pi_1 \times \pi_2 = i_{P(a,b)}^{G_{a+b}}(\pi_1 \boxtimes \pi_2)$$

for normalized parabolic induction. A rank-zero representation is a vector space, and its product with a representation means tensoring with that vector space.

Theorem 4.19 (Leibniz filtration). *Let π_i be smooth representations of G_{n_i} . For every k , the derivative $(\pi_1 \times \cdots \times \pi_r)^{(k)}$ has a finite filtration whose successive quotients are*

$$\pi_1^{(k_1)} \times \cdots \times \pi_r^{(k_r)}, \quad 0 \leq k_i \leq n_i, \quad \sum_i k_i = k.$$

Terms that are zero are omitted. No further modulus twists occur with the normalized conventions fixed above.

Proof. For two factors this is the derivative form of the geometric lemma: filter the restriction of the induced representation to the mirabolic subgroup by its orbits on the flag variety. After taking the twisted quotient, a stratum contributes the term in which k_1 coordinates come from the first block and k_2 from the second. Its half-modulus factors are exactly those already present in normalized derivatives and normalized induction. This is [BZ, Lemma 4.5]. Applying this two-factor statement repeatedly proves the formula for r factors. $\square$

For finite-length input, let $[\sigma]$ denote the class of σ in the Grothendieck group. The theorem gives

$$[(\pi_1 \times \cdots \times \pi_r)^{(k)}] = \sum_{k_1 + \cdots + k_r = k} [\pi_1^{(k_1)} \times \cdots \times \pi_r^{(k_r)}].$$

This is a statement about composition factors with multiplicities; it does not assert that the filtration splits. Equivalently, on the graded representation ring,

$$\mathcal{D}_t[\pi] = \sum_{k=0}^n [\pi^{(k)}] t^k, \quad \mathcal{D}_t([\pi_1] \times [\pi_2]) = \mathcal{D}_t[\pi_1] \times \mathcal{D}_t[\pi_2].$$

The coefficient of t is the usual first-order Leibniz identity.

Corollary 4.20. *For nonzero finite-length π_1, π_2 , put $d_i = d(\pi_i)$. Then*

$$d(\pi_1 \times \pi_2) = d_1 + d_2, \quad (\pi_1 \times \pi_2)^{(d_1+d_2)} \simeq \pi_1^{(d_1)} \times \pi_2^{(d_2)}.$$

If both factors are irreducible and generic, their induced product has exactly one generic irreducible composition factor, with multiplicity one.

Proof. In degree $d_1 + d_2$ only the term (d_1, d_2) survives in the filtration, and its parabolic induction is nonzero. Higher degrees vanish. If both factors are generic, this top derivative is $\mathbb{C}$. Exactness of the top derivative and Whittaker uniqueness then count the generic factors of a composition series. $\square$

4.1.7. *Characters, principal series, and supercuspidal representations.* Let χ be a smooth character of $F^\times$. We use $\chi \circ \det$ for its inflation to any positive rank. Twisting commutes with derivatives:

$$((\chi \circ \det) \otimes \pi)^{(k)} \simeq (\chi \circ \det) \otimes \pi^{(k)},$$

where the twist on the right is on $\mathrm{GL}_{n-k}(F)$ and is trivial at rank zero. This follows since determinant characters are trivial on every unipotent group in the definition.

Example 4.21 (Characters). For χ on G_1 , restriction to $\mathcal{P}_1 = \{1\}$ gives $\chi^{(1)} = \mathbb{C}$. For $n \geq 2$ and $\pi = \chi \circ \det$ on G_n ,

$$\pi^{(1)} = (\chi \circ \det) \nu_{n-1}^{-1/2}, \quad \pi^{(k)} = 0 \quad (2 \leq k \leq n).$$

Indeed V_n acts trivially, so $\Phi_n^- \pi = 0$, while $\Psi_n^- \pi$ is its restriction with the half-modulus twist. In particular, a one-dimensional representation of G_n is not generic when $n > 1$.

Example 4.22 (Principal series of G_2 and G_3). Let $I_2 = \chi_1 \times \chi_2$ be normalized induction from the upper Borel of G_2 . The Leibniz filtration gives

$$\mathrm{gr} I_2^{(1)} = \chi_1 \oplus \chi_2, \quad I_2^{(2)} = \mathbb{C}.$$

For $I_3 = \chi_1 \times \chi_2 \times \chi_3$,

$$\begin{aligned} \mathrm{gr} I_3^{(1)} &= (\chi_1 \times \chi_2) \oplus (\chi_1 \times \chi_3) \oplus (\chi_2 \times \chi_3), \\ \mathrm{gr} I_3^{(2)} &= \chi_1 \oplus \chi_2 \oplus \chi_3, \quad I_3^{(3)} = \mathbb{C}. \end{aligned}$$

Here gr refers to the indicated derivative filtration. For a principal series of G_n , its k th derivative has graded pieces given by the principal series obtained by deleting k of the characters. Hence

$$\dim I_n^{(n-1)} = n, \quad \dim I_n^{(n)} = 1.$$

The formulas hold even if the induced representation is reducible. They describe its entire derivative, not automatically the derivative of each irreducible constituent.

Theorem 4.23 (Supercuspidal derivatives). *If ρ is an irreducible supercuspidal representation of G_d , then*

$$\rho^{(k)} = 0 \quad (0 < k < d), \quad \rho^{(d)} = \mathbb{C}.$$

Conversely, an irreducible smooth representation of G_d with all these intermediate derivatives zero is supercuspidal.

Proof. The vanishing follows from Proposition 4.13 and the vanishing of proper Jacquet modules. Every irreducible supercuspidal representation of a general linear group is generic; Whittaker uniqueness gives the one-dimensional top derivative. The converse is the cuspidality criterion stated after the theorem in [BZ, Section 4.4]. $\square$

The mirabolic filtration now has a single nonzero layer, so

$$\rho|_{\mathcal{P}_d} \simeq (\Phi^+)^{d-1} \Psi_1^+(\mathbb{C}).$$

It is irreducible by Corollary 4.10. Irreducibility of a mirabolic restriction alone does not characterize supercuspidality: the preceding character example also restricts irreducibly.

Example 4.24 (A mixed induced representation of G_3). For supercuspidal ρ of G_2 and a character χ of G_1 , the normalized induction $J = \rho \times \chi$ satisfies

$$J^{(1)} \simeq \rho, \quad J^{(2)} \simeq \chi, \quad J^{(3)} \simeq \mathbb{C}.$$

For each order there is only one nonzero term in the Leibniz filtration, so these are isomorphisms, not just identities of classes.

Remark 4.25 (Higher derivatives are not iterated first derivatives). For $\pi = \mathbf{1}_{G_3}$ one has

$$\pi^{(1)} = \nu_2^{-1/2}, \quad (\pi^{(1)})^{(1)} = \nu_1^{-1}, \quad \pi^{(2)} = 0.$$

The definition of $\pi^{(2)}$ uses $\Psi^- \Phi^-$, while the iteration on the left uses Ψ^- twice after restriction to a smaller general linear group. These impose different characters on the unipotent subgroups.

4.1.8. Segments and generalized Steinberg representations. Fix an irreducible supercuspidal representation ρ of G_d . Write $\nu^s \rho$ for $\nu_d^s \otimes \rho$. A *segment* is

$$\Delta = [\nu^a \rho, \nu^b \rho] = \{\nu^a \rho, \nu^{a+1} \rho, \dots, \nu^b \rho\}, \quad \ell = b - a + 1 \in \mathbb{Z}_{>0}.$$

Here a, b may be real, with integral difference. There are two representations of $G_{d\ell}$ that must be distinguished:

notation	construction from $\nu^a \rho \times \nu^{a+1} \rho \times \dots \times \nu^b \rho$
$Z(\Delta)$	unique irreducible subrepresentation
$\delta(\Delta)$	unique irreducible quotient.

These uniqueness statements are part of the segment classification. The representation $\delta(\Delta)$ is essentially square-integrable; it is also the unique irreducible subrepresentation of the product in descending order. The original notation $\langle \Delta \rangle$ in [Zel] is $Z(\Delta)$. Set $Z(\emptyset) = \delta(\emptyset) = \mathbb{C}$ in rank zero.

Theorem 4.26 (Segment derivatives). *For the right derivatives fixed above,*

$$Z(\Delta)^{(d)} \simeq Z([\nu^a \rho, \nu^{b-1} \rho]), \quad Z(\Delta)^{(j)} = 0 \quad (j > 0, j \neq d).$$

For the essentially square-integrable representation,

$$\delta(\Delta)^{(rd)} \simeq \delta([\nu^{a+r} \rho, \nu^b \rho]), \quad 0 \leq r \leq \ell,$$

and $\delta(\Delta)^{(j)} = 0$ when $d \nmid j$. For $r = \ell$, the segment on the right is empty.

Proof. The assertion about $Z(\Delta)$ is [Zel, Theorem 3.5]. For $\delta(\Delta)$, the standard Jacquet formula is

$$\begin{aligned} r_{(d(\ell-r), dr)} \delta(\Delta) &\simeq \delta([\nu^{a+r} \rho, \nu^b \rho]) \\ &\boxtimes \delta([\nu^a \rho, \nu^{a+r-1} \rho]), \quad 0 < r < \ell, \end{aligned}$$

and the corresponding Jacquet module is zero for a second block size not divisible by d ; see [Zel, Proposition 9.5]. Each essentially square-integrable segment representation is generic. Taking the Whittaker quotient of the second factor and using Proposition 4.13 gives the derivative formula. The top case follows from genericity and uniqueness. This is also [Zel, Proposition 9.6]. $\square$

Thus $Z(\Delta)^{(d)}$ deletes the upper endpoint, whereas the derivative $\delta(\Delta)^{(rd)}$ removes the r entries at the lower endpoint. Both agree with $\rho^{(d)} = \mathbb{C}$ for a segment of length one.

Corollary 4.27 (Steinberg derivatives). *Let*

$$\text{St}_\ell(\rho) = \delta([\nu^{-(\ell-1)/2} \rho, \nu^{(\ell-1)/2} \rho]).$$

For $0 \leq r < \ell$,

$$\text{St}_\ell(\rho)^{(rd)} \simeq \nu_{d(\ell-r)}^{r/2} \text{St}_{\ell-r}(\rho), \quad \text{St}_\ell(\rho)^{(d\ell)} = \mathbb{C},$$

and the remaining positive derivatives vanish. In particular, for the ordinary Steinberg representation of G_n ,

$$\text{St}_n^{(k)} \simeq \nu_{n-k}^{k/2} \text{St}_{n-k} \quad (0 \leq k < n), \quad \text{St}_n^{(n)} = \mathbb{C}.$$

Proof. After deleting r entries at the lower endpoint, the remaining segment has length $\ell - r$ and center $r/2$. Twisting the symmetric segment of that length by $\nu^{r/2}$ gives exactly these endpoints. $\square$

Example 4.28. For G_2 ,

$$Z([\nu^{-1/2}, \nu^{1/2}]) = \mathbf{1}_{G_2}, \quad \delta([\nu^{-1/2}, \nu^{1/2}]) = \text{St}_2.$$

The first derivatives are respectively $\nu_1^{-1/2}$ and $\nu_1^{1/2}$; the second derivatives are 0 and $\mathbb{C}$. For G_3 the Steinberg formulas read

$$\text{St}_3^{(1)} = \nu_2^{1/2} \text{St}_2, \quad \text{St}_3^{(2)} = \nu_1, \quad \text{St}_3^{(3)} = \mathbb{C}.$$

These also check the normalization against the character and principal-series calculations above.

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