

NOTES ON LOCAL THETA CORRESPONDENCE

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1. INTRODUCTION

These notes give an exposition of Weil's construction of the Weil representation, with particular attention to the Heisenberg group, Schrödinger models, and the local theta correspondence. We supply the linear algebra and intertwining calculations needed to pass from the Heisenberg representation to the metaplectic group. For the local representation-theoretic viewpoint, our main reference is [Kud, Chapter I].

The initial construction is projective: it gives a homomorphism into $U(L^2(X))/\mathbb{S}^1$, rather than an ordinary unitary representation of the symplectic group. The metaplectic extension makes this projective action linear. We usually write the operators on the Bruhat–Schwartz space $\mathcal{S}(X)$, where the formulas are explicit. This is an invariant dense subspace of $L^2(X)$, and the suitably normalized operators extend to unitary operators on its Hilbert completion. In the non-Archimedean case we also use the smooth algebraic representation on $\mathcal{S}(X)$; in the Archimedean case the uniqueness statements below are understood in the unitary Hilbert-space category.

2. THE HEISENBERG GROUP AND THE HEISENBERG REPRESENTATION

Let k be a local field of characteristic zero: $\mathbb{R}$, $\mathbb{C}$, or a finite extension of $\mathbb{Q}_p$. Fix a symplectic k -vector space $(W, \langle \cdot, \cdot \rangle)$ of dimension $2n$ and a nontrivial continuous unitary additive character $\psi : k \rightarrow \mathbb{S}^1$. For a finite-dimensional k -vector space V , $\mathcal{S}(V)$ denotes the usual Schwartz space in the Archimedean case and $C_c^\infty(V)$ in the non-Archimedean case. All vectors are columns, and matrices act on the left.

2.1. The Heisenberg group as a central extension.

Definition 2.1. For a subspace $A \subseteq W$, put

$$A^\perp = \{w \in W : \langle w, a \rangle = 0 \text{ for every } a \in A\}.$$

The subspace is *isotropic* if $A \subseteq A^\perp$ and *Lagrangian* if $A = A^\perp$. Equivalently, a Lagrangian is an isotropic subspace of dimension n . A *polarization* is a decomposition $W = X \oplus Y$ into two Lagrangian subspaces.

The nondegeneracy of the form gives $\dim A + \dim A^\perp = 2n$. Every isotropic subspace extends to a Lagrangian, and every Lagrangian has a Lagrangian complement: extend a basis of it to a symplectic basis.

Definition 2.2 (Heisenberg group). The Heisenberg group is $H(W) = W \times k$ with multiplication

$$(w_1, t_1)(w_2, t_2) = (w_1 + w_2, t_1 + t_2 + \tfrac{1}{2}\langle w_1, w_2 \rangle).$$

Bilinearity proves associativity. The identity is $(0, 0)$, inversion is $(w, t)^{-1} = (-w, -t)$, and

$$[(w_1, t_1), (w_2, t_2)] = (0, \langle w_1, w_2 \rangle).$$

Consequently the center is $Z(W) = \{(0, t) : t \in k\}$, and there is a central extension

$$1 \longrightarrow (k, +) \longrightarrow H(W) \longrightarrow (W, +) \longrightarrow 1.$$

We also write ψ for the character $(0, t) \mapsto \psi(t)$ of $Z(W)$.

Theorem 2.3 (Stone–von Neumann). *Up to unitary equivalence, there is a unique strongly continuous irreducible unitary representation of $H(W)$ with central character ψ . If k is non-Archimedean, there is likewise a unique irreducible smooth complex representation with that central character.*

The corresponding representation is called the *Heisenberg representation*. The smooth non-Archimedean statement and its induced models are discussed in [Kud, Sections I.1–I.2]. Specifying the central character alone does not imply irreducibility; that condition must be checked when constructing a model.

2.2. Models of the Heisenberg representation. Put

$$Z_\psi(W) = \{(0, t) : \psi(t) = 1\}, \quad A^{\perp, \psi} = \{w : \psi(\langle w, a \rangle) = 1 \text{ for all } a \in A\}.$$

For a k -linear subspace A , one has $A^{\perp, \psi} = A^\perp$: a nonzero linear map $A \rightarrow k$ is surjective, and its image cannot be contained in $\ker \psi$. Thus the self-dual k -subspaces, in the sense $A = A^{\perp, \psi}$, are precisely the Lagrangians.

Lemma 2.4. *If A is Lagrangian and $H(A) = A \times k$, then $H(A)/Z_\psi(W)$ is a maximal abelian subgroup of $H(W)/Z_\psi(W)$.*

Proof. It is abelian because $\langle A, A \rangle = 0$. An element (w, t) commutes with it modulo $Z_\psi(W)$ precisely when $\psi(\langle w, a \rangle) = 1$ for all $a \in A$, or equivalently $w \in A^{\perp, \psi} = A$. $\square$

Corollary 2.5. *The central character extends to $\psi_A : H(A) \rightarrow \mathbb{S}^1$ by $\psi_A(a, t) = \psi(t)$.*

Proof. The alternating form vanishes on A , so multiplication in $H(A)$ is addition in $A \times k$. $\square$

Choose a Lagrangian complement B to A . Let $\mathcal{H}_A$ be the space of functions $f : H(W) \rightarrow \mathbb{C}$ satisfying

$$f(h_A h) = \psi_A(h_A) f(h), \quad b \mapsto f((b, 0)) \in \mathcal{S}(B).$$

Restriction to B identifies this space with $\mathcal{S}(B)$. Right translation defines

$$(\rho_A(h_0) f)(h) = f(h h_0).$$

In the non-Archimedean case this is compact smooth induction,

$$\mathcal{H}_A = \text{c-Ind}_{H(A)}^{H(W)} \psi_A.$$

For Archimedean k we use the displayed Schwartz model and its $L^2(B)$ completion, rather than this compact-support description. Changing the complementary Lagrangian changes the coordinates by a linear map and a quadratic phase, preserving the Schwartz condition.

Definition 2.6. The representation $(\rho_A, \mathcal{H}_A)$ is the *induced model* associated with A . Its Hilbert completion is formed using a Haar measure on $H(A) \backslash H(W) \simeq B$.

The Schrödinger realization below proves directly that these models are irreducible in the relevant category. Theorem 2.3 then identifies all of them with the Heisenberg representation.

There are two useful operations on the models. For $g \in \text{Sp}(W)$, let $g \cdot (w, t) = (gw, t)$ and put

$$T_g : \mathcal{H}_A \rightarrow \mathcal{H}_{gA}, \quad (T_g f)(h) = f(g^{-1} \cdot h).$$

Its covariance is $T_g \rho_A(h) = \rho_{gA}(g \cdot h) T_g$. For Lagrangians A, B , choose a Haar measure on $A/(A \cap B)$ and set

$$I_{B,A} : \mathcal{H}_B \rightarrow \mathcal{H}_A, \quad (I_{B,A} f)(h) = \int_{A/(A \cap B)} f((a, 0)h) da.$$

The integrand descends to the quotient, since A is isotropic and ψ_B is trivial on $(A \cap B) \times \{0\}$. Right translation commutes with this integral, so $I_{B,A} \rho_B(h) = \rho_A(h) I_{B,A}$. In coordinates adapted to $A \cap B$, this operator is a partial Fourier transform, up to linear changes of variables and quadratic phases. Hence it is nonzero, preserves Schwartz spaces, and extends after positive normalization to a unitary intertwiner. This also justifies convergence of the integral on the stated test spaces.

Consequently

$$\Omega_A(g) = I_{gA,A} T_g$$

is an invertible operator on $\mathcal{H}_A$ satisfying

$$\Omega_A(g)\rho_A(h)\Omega_A(g)^{-1} = \rho_A(g \cdot h).$$

Normalize each operator to be unitary on the Hilbert completion. Unitary Schur's lemma then gives scalars $c_A(g_1, g_2) \in \mathbb{S}^1$ such that

$$\Omega_A(g_1)\Omega_A(g_2) = c_A(g_1, g_2)\Omega_A(g_1g_2).$$

Thus the symplectic action gives a projective representation. In particular, the cocycle convention here always has the product of the two operators on the left of this equality.

2.2.1. The Schrödinger model. Fix $W = X \oplus Y$. Our main realization is induced from $H(Y)$ and acts on $\mathcal{S}(X)$. The map

$$\Phi_Y : \mathcal{H}_Y \longrightarrow \mathcal{S}(X), \quad \Phi_Y(f)(z) = f((z, 0)),$$

is an isomorphism of vector spaces.

Theorem 2.7. *The formula*

$$(\sigma_\psi(x + y, t)\phi)(z) = \psi\left(t + \langle z, y \rangle + \frac{1}{2}\langle x, y \rangle\right)\phi(z + x), \quad x, z \in X, \ y \in Y,$$

defines a representation of $H(W)$ on $\mathcal{S}(X)$, isomorphic to ρ_Y . It extends to an irreducible unitary representation on $L^2(X)$. For non-Archimedean k , its representation on $\mathcal{S}(X)$ is smooth and irreducible.

Proof. The identity

$$(z, 0)(x + y, t) = (y, t + \langle z, y \rangle + \frac{1}{2}\langle x, y \rangle)(z + x, 0)$$

proves the formula by left $H(Y)$ -covariance. Alternatively, multiplying two displayed operators recovers the Heisenberg group law. Translation and multiplication by unitary characters preserve the L^2 norm, giving the strongly continuous unitary extension.

For unitary irreducibility, a bounded operator commuting with all modulations is a multiplication operator. Commuting also with all translations forces its multiplier to be constant almost everywhere. Thus the commutant consists of scalars, which proves irreducibility. In the non-Archimedean Schwartz space, Fourier averaging over compact groups of modulations cuts a nonzero function down to a compact open coset on which it is constant. Translations and further such cut-offs then generate the characteristic functions of all compact open cosets, which span $\mathcal{S}(X)$. This proves smooth irreducibility. $\square$

The elementary actions are therefore

$$\sigma_\psi(x, 0)\phi(z) = \phi(z + x), \quad \sigma_\psi(y, 0)\phi(z) = \psi(\langle z, y \rangle)\phi(z), \quad \sigma_\psi(0, t) = \psi(t).$$

They also fix the signs in every later intertwining formula.

Choose dual bases e_i of X and f_i of Y with $\langle e_i, f_j \rangle = \delta_{ij}$. The self-dual Haar measure on k for ψ gives product measures dx and dy for which

$$\widehat{\phi}(y) = \int_X \phi(x)\psi(\langle x, y \rangle) dx, \quad \phi(x) = \int_Y \widehat{\phi}(y)\psi(-\langle x, y \rangle) dy.$$

Fourier transform preserves Schwartz spaces and is unitary on L^2 . Equivalently, given one Haar measure on X , these identities determine its dual measure on Y ; the pair is what matters intrinsically. Induction from $H(X)$ instead gives a model on $\mathcal{S}(Y)$ with

$$(\sigma_{Y,\psi}(x + y, t)\phi)(y_0) = \psi\left(t + \langle y_0, x \rangle + \frac{1}{2}\langle y, x \rangle\right)\phi(y_0 + y).$$

The Fourier transform just defined intertwines these two models.

2.3. Symplectic groups. The bicharacter $(w, w') \mapsto \psi(\langle w, w' \rangle)$ identifies W with its Pontryagin dual. In the symplectic basis fixed above,

$$w = \begin{pmatrix} x \\ y \end{pmatrix}, \quad J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \quad \langle w, w' \rangle = w^t J w'.$$

Thus

$$\mathrm{Sp}(W) = \{g \in \mathrm{GL}_{2n}(k) : g^t J g = J\}.$$

Writing $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, the blocks have types $\alpha : X \rightarrow X$, $\beta : Y \rightarrow X$, $\gamma : X \rightarrow Y$, and $\delta : Y \rightarrow Y$.

The symplectic relations are

$$\alpha^t \gamma = \gamma^t \alpha, \quad \beta^t \delta = \delta^t \beta, \quad \alpha^t \delta - \gamma^t \beta = I_n.$$

We use the stabilizer of Y ,

$$P = P(Y) = \{g \in \mathrm{Sp}(W) : gY = Y\}.$$

It is the *lower* block triangular subgroup, so $\beta = 0$. The preceding relations give $\delta = \alpha^{-t}$ and $\gamma \alpha^{-1}$ symmetric. Define

$$m(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-t} \end{pmatrix}, \quad n(c) = \begin{pmatrix} I_n & 0 \\ c & I_n \end{pmatrix}, \quad a \in \mathrm{GL}(X), \quad c = c^t.$$

Here $c : X \rightarrow Y$ is symmetric in the intrinsic sense $\langle x_1, cx_2 \rangle = \langle x_2, cx_1 \rangle$.

Lemma 2.8. *The symplectic group acts transitively on the Lagrangian Grassmannian*

$$\mathrm{LGr}(n, W) = \{L \in \mathrm{Gr}(n, W) : \langle L, L \rangle = 0\}.$$

Proof. Extend bases of two Lagrangians to symplectic bases of W . The linear map carrying one symplectic basis to the other is symplectic and carries the first Lagrangian to the second. $\square$

Proposition 2.9. *The stabilizer $P(Y)$ is a parabolic subgroup of $\mathrm{Sp}(W)$.*

Proof. The orbit map identifies the algebraic homogeneous variety $\mathrm{Sp}(W)/P(Y)$ with $\mathrm{LGr}(n, W)$ by $gP(Y) \mapsto gY$. Isotropy is a closed condition on the Grassmannian. Since the Grassmannian is projective by its Plücker embedding in $\mathbb{P}(\bigwedge^n W)$, the Lagrangian Grassmannian is projective as well. Hence $P(Y)$ is parabolic. The same orbit identification holds on k -points by the preceding basis argument. $\square$

Definition 2.10. The subgroup $P(Y)$ is the *Siegel parabolic* associated with Y . Its *Siegel Levi subgroup* and *unipotent radical* are

$$M = \{m(a) : a \in \mathrm{GL}(X)\}, \quad N = \{n(c) : c = c^t\}.$$

Every element of P has the unique factorization

$$\begin{pmatrix} a & 0 \\ \gamma & a^{-t} \end{pmatrix} = n(\gamma a^{-1})m(a).$$

Moreover,

$$m(a)n(c)m(a)^{-1} = n(a^{-t}ca^{-1}).$$

Thus $P = MN = NM$, $M \cap N = 1$, and $1 \rightarrow N \rightarrow P \rightarrow M \rightarrow 1$ is split. The order of these factors will be kept explicit in operator calculations.

2.3.1. Generalized Bruhat decomposition. For $0 \leq j \leq n$, write

$$X = X_0 \oplus X_1, \quad Y = Y_0 \oplus Y_1,$$

where X_0, Y_0 are spanned by the first $n-j$ basis vectors and X_1, Y_1 by the last j . Let w_j fix $X_0 \oplus Y_0$ and satisfy

$$w_j e_i = -f_i, \quad w_j f_i = e_i \quad (n-j < i \leq n).$$

In particular $w_0 = 1$, $w_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$, and $w_j Y = Y_0 \oplus X_1$.

Theorem 2.11 (Generalized Bruhat decomposition). *There is a disjoint decomposition*

$$\mathrm{Sp}(W) = \bigsqcup_{j=0}^n Pw_jP.$$

More precisely, for $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$,

$$g \in Pw_jP \iff \dim(Y \cap gY) = n - j \iff \mathrm{rank} \beta = j.$$

Proof. Double cosets correspond to P -orbits on the Lagrangian Grassmannian. Let L be Lagrangian with $\dim(L \cap Y) = n - j$. The Levi subgroup acts as the full general linear group on Y , so we may arrange $L \cap Y = Y_0$. Then $L \subseteq Y_0^\perp = X_1 \oplus Y$. In $Y_0^\perp/Y_0 \simeq X_1 \oplus Y_1$, its image is transverse to Y_1 , hence is the graph of a linear map $c : X_1 \rightarrow Y_1$. Consequently

$$L = Y_0 \oplus \{x + cx : x \in X_1\}.$$

Isotropy gives $\langle x, cx' \rangle = \langle x', cx \rangle$, so c is symmetric. Extend c by zero on X_0 . The element $n(-c) \in P$ sends L to $Y_0 \oplus X_1 = w_jY$. Conversely, the intersection dimension is P -invariant, so different j give different orbits. Finally the map $Y \rightarrow X$, $y \mapsto \beta y$, is the projection of gY to X . Its kernel has dimension $\dim(Y \cap gY)$, proving the rank criterion. $\square$

The open cell is the locus where β is invertible. With the column convention and the choice $P = P(Y)$, its rank is determined by β throughout.

3. THE METAPLECTIC GROUP

3.1. The metaplectic group as a central extension. Fix the central character ψ and let $\mathcal{L}_Y = L^2(X)$ be the Hilbert completion of the induced model $\mathcal{H}_Y \simeq \mathcal{S}(X)$. For $g \in \mathrm{Sp}(W)$, the representation $h \mapsto \sigma_\psi(g \cdot h)$ has the same central character as σ_ψ . By Stone–von Neumann, there is a unitary operator $R(g)$ such that

$$R(g)\sigma_\psi(h)R(g)^{-1} = \sigma_\psi(g \cdot h).$$

Unitary Schur's lemma makes $R(g)$ unique up to a scalar in $\mathbb{S}^1$. Consequently $g \mapsto [R(g)]$ is a projective unitary representation of $\mathrm{Sp}(W)$.

Definition 3.1. The *unitary scalar extension* $\mathrm{Mp}_\psi^{\mathbb{S}^1}(W)$ consists of pairs $(g, T) \in \mathrm{Sp}(W) \times \mathrm{U}(\mathcal{L}_Y)$ such that

$$T\sigma_\psi(h)T^{-1} = \sigma_\psi(g \cdot h) \quad \text{for every } h \in H(W).$$

Its multiplication is $(g_1, T_1)(g_2, T_2) = (g_1g_2, T_1T_2)$, giving

$$1 \longrightarrow \mathbb{S}^1 \longrightarrow \mathrm{Mp}_\psi^{\mathbb{S}^1}(W) \longrightarrow \mathrm{Sp}(W) \longrightarrow 1.$$

The *complex scalar extension* is its pushout along $\mathbb{S}^1 \hookrightarrow \mathbb{C}^\times$, denoted $\mathrm{Mp}_\psi^{\mathbb{C}^\times}(W)$.

The latter can also be described using all nonzero complex multiples of the unitary implementing operators. These operators preserve the Schwartz model, as the formulas below show. Conjugation by an equivalence of Heisenberg models identifies their scalar extensions; its scalar ambiguity cancels in conjugation. Thus the construction for fixed ψ does not depend on the polarization used to realize it.

Definition 3.2. The *Weil representation* on either scalar extension is the tautological representation

$$\omega_\psi(g, T) = T.$$

In particular, a central scalar z acts as $z \mathrm{Id}$.

These scalar extensions have kernels $\mathbb{S}^1$ and $\mathbb{C}^\times$, respectively. The usual metaplectic *double cover*, whose kernel is $\{\pm 1\}$, requires the additional reduction in the next subsection on the two-fold cover. The Weil representation on that double cover is genuine: its nontrivial central element acts by $-\mathrm{Id}$. Over a non-Archimedean field its Schwartz realization is a smooth representation of the double cover; over every local field considered here its Hilbert completion is unitary.

3.2. The Leray cocycle. For each Lagrangian A , fix a Haar measure on W/A and use it to form the Hilbert completion $\mathcal{L}_A$ of $\mathcal{H}_A$. Retain the source-first notation of the induced models:

$$I_{B,A} : \mathcal{H}_B \longrightarrow \mathcal{H}_A, \quad (I_{B,A}f)(h) = \int_{A/(A \cap B)} f((a, 0)h) d\dot{a}.$$

Lemma 3.3. *There is a unique positive normalization of the measure in this integral for which the operator is unitary. Denote the resulting operator by $A_{B,A}$. It preserves Schwartz spaces and satisfies*

$$A_{A,A} = 1, \quad A_{B,A}^{-1} = A_{A,B}.$$

Proof. Choose symplectic coordinates adapted to $A \cap B$. The operator is a partial Fourier transform in the directions where A and B differ, and the identity in the remaining directions, up to linear changes and unitary quadratic phases. Plancherel gives a positive normalization making it unitary. Two Haar measures differ by a positive scalar, so this normalization is unique. The reverse operator uses the dual Fourier transform, giving the inverse identity. $\square$

The transport $T_g f(h) = f(g^{-1} \cdot h)$ also needs its positive normalization when the quotient measures are fixed. Let $U_{g,A}$ be that unitary normalization, from $\mathcal{L}_A$ to $\mathcal{L}_{gA}$. More explicitly, if $g_*\mu_A = j(g, A)\mu_{gA}$ for the chosen quotient measures, then

$$U_{g,A} = j(g, A)^{1/2} T_g.$$

The chain rule for measure transport and uniqueness of positive normalization give

$$\begin{aligned} U_{g_1, g_2 A} U_{g_2, A} &= U_{g_1 g_2, A}, \\ U_{g, A} A_{B, A} &= A_{gB, gA} U_{g, B}. \end{aligned}$$

These identities also hold on the Schwartz models.

Fix $A = Y$, and define

$$\Omega_Y(g) = A_{gY, Y} U_{g, Y}, \quad r_Y(g) = \Phi_Y \Omega_Y(g) \Phi_Y^{-1}.$$

They implement the same covariance:

$$r_Y(g) \sigma_\psi(h) r_Y(g)^{-1} = \sigma_\psi(g \cdot h).$$

Definition 3.4. The *Leray cocycle* associated with Y and ψ is defined by

$$r_Y(g_1) r_Y(g_2) = c_Y(g_1, g_2) r_Y(g_1 g_2).$$

Here $c_Y(g_1, g_2) \in \mathbb{S}^1$ and $r_Y(1) = 1$.

Associativity of operators gives

$$c_Y(g_1, g_2) c_Y(g_1 g_2, g_3) = c_Y(g_1, g_2 g_3) c_Y(g_2, g_3), \quad c_Y(1, g) = c_Y(g, 1) = 1.$$

Thus, set-theoretically, the scalar extension is $\mathrm{Sp}(W) \times \mathbb{C}^\times$, with multiplication

$$(g_1, z_1)(g_2, z_2) = (g_1 g_2, z_1 z_2 c_Y(g_1, g_2)), \quad \omega_\psi(g, z) = z r_Y(g).$$

Replacing the section by $r'_Y(g) = b(g) r_Y(g)$ changes its cocycle to

$$c'_Y(g_1, g_2) = \frac{b(g_1) b(g_2)}{b(g_1 g_2)} c_Y(g_1, g_2).$$

Proposition 3.5. *For $p_1, p_2 \in P(Y)$ and $g \in \mathrm{Sp}(W)$,*

$$r_Y(p_1 g p_2) = r_Y(p_1) r_Y(g) r_Y(p_2).$$

In particular r_Y restricts to a group homomorphism on $P(Y)$, and $c_Y(p, g) = c_Y(g, p) = 1$ for $p \in P(Y)$.

Proof. If $pY = Y$, then $\Omega_Y(p) = U_{p, Y}$. Apply the two transport identities above to the definition of $\Omega_Y(p_1 g p_2)$ and transfer the result by Φ_Y . $\square$

In the fixed column coordinates this normalization gives

$$\begin{aligned}(r_Y(m(a))\phi)(z) &= |\det a|^{-1/2} \phi(a^{-1}z), \\ (r_Y(n(c))\phi)(z) &= \psi(-\tfrac{1}{2}z^t c z) \phi(z), \\ (r_Y(w)\phi)(z) &= \int_X \psi(z^t x) \phi(x) dx,\end{aligned}$$

where dx is self-dual for ψ and $w = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. For example, $m(a)$ transports dx to $|\det a|^{-1} dx$, explaining the factor in the first formula. The second follows by evaluating $T_{n(c)}f$ at $(z, 0)$, and the third is the normalized intertwiner between the two transverse Lagrangian models. These formulas fix the sign of the Fourier transform used below.

Corollary 3.6. *For $p_1, p_2, p \in P(Y)$,*

$$c_Y(p_1 g_1 p^{-1}, p g_2 p_2) = c_Y(g_1, g_2).$$

Proof. Use Proposition 3.5 in the defining product of the two operators; the adjacent factors $r_Y(p)^{-1} r_Y(p)$ cancel. $\square$

3.3. The Weil index. Here a quadratic form is symmetric data independent of the alternating form on W . Let $E = k^d$ and

$$q(x) = \tfrac{1}{2} x^t A x, \quad A = A^t \in \mathrm{GL}_d(k).$$

With self-dual measure, use the positive Fourier transform $\widehat{f}(\xi) = \int_E f(x) \psi(x^t \xi) dx$. In particular, $\widehat{\widehat{f}}(x) = f(-x)$.

Definition 3.7. The *Weil index* $\Gamma_\psi(q) \in \mathbb{S}^1$ is the scalar in the Gaussian Fourier identity

$$\mathcal{F}_\psi(\psi(q))(\xi) = \Gamma_\psi(q) |\det A|^{-1/2} \psi(-\tfrac{1}{2} \xi^t A^{-1} \xi).$$

The equality is one of tempered distributions, or equivalently of oscillatory integrals, rather than an assertion of absolute convergence. For a degenerate form, define its index using the nondegenerate form on $E/\mathrm{rad}(q)$, and put $\Gamma_\psi(0) = 1$.

Completing the square explains the quadratic term on the right. The remaining constant has modulus one by Fourier inversion and Plancherel. In particular the determinant here is that of the polar form A ; if $q = \sum a_i x_i^2$, then $A = 2 \mathrm{diag}(a_i)$.

The Weil index is invariant under isometry and multiplicative under orthogonal direct sums. Moreover,

$$\Gamma_\psi(-q) = \overline{\Gamma_\psi(q)}, \quad \Gamma_\psi(H) = 1$$

for a hyperbolic plane H . The last identity follows by integrating $\psi(xy)$ as a Fourier kernel. Thus the index depends only on the Witt class of the nondegenerate form.

For $a \in k^\times$, put

$$\gamma_\psi(a) = \frac{\Gamma_\psi(ax^2)}{\Gamma_\psi(x^2)}.$$

The basic one-dimensional identities are

$$\begin{aligned}\gamma_\psi(ab^2) &= \gamma_\psi(a), & \gamma_\psi(ab) &= (a, b)_k \gamma_\psi(a) \gamma_\psi(b), \\ \gamma_\psi(a)^2 &= (a, -1)_k = (a, a)_k, & \gamma_\psi(a)^4 &= 1, \\ \Gamma_\psi(x^2)^2 &= \gamma_\psi(-1)^{-1}, & \Gamma_\psi(x^2)^8 &= 1.\end{aligned}$$

The Hilbert symbol $(a, b)_k$ equals 1 precisely when the quaternion algebra (a, b) is split. For nonsquare b , this is equivalent to a being a norm from $k(\sqrt{b})$; for square b it is always 1. The product identity is the Weil-index/Hilbert-symbol relation; the other displayed identities follow from it, square invariance, and $\Gamma_\psi(-x^2) = \Gamma_\psi(x^2)^{-1}$.

Consequently, for a nondegenerate diagonal form $q = a_1 x_1^2 + \cdots + a_d x_d^2$,

$$\Gamma_\psi(q) = \Gamma_\psi(x^2)^d \gamma_\psi(\det q) \epsilon(q), \quad \epsilon(q) = \prod_{i < j} (a_i, a_j)_k,$$

where $\det q = \prod_i a_i$ modulo squares. Explicit local formulas and the normalization of the Gaussian integral are recorded in the appendix; compare [Kud, Chapter I, Section 4] and [Rao, Appendix].

3.4. The cocycle condition. We now compute the scalar in the product of normalized intertwiners. For an ordered triple of Lagrangians (L_0, L_1, L_2) , form

$$K = \{(x_0, x_1, x_2) \in L_0 \oplus L_1 \oplus L_2 : x_0 + x_1 + x_2 = 0\}, \quad q_K(x_0, x_1, x_2) = -\frac{1}{2}\langle x_0, x_1 \rangle.$$

Denote the induced nondegenerate form on $K/\text{rad}(q_K)$ by $L(L_0, L_1, L_2)$, the *Leray* or *Maslov form*. The appendix gives its linear algebra and reduction properties. The order of the three Lagrangians, and the minus sign in this definition, are part of our convention.

Lemma 3.8 (Composition of Lagrangian intertwiners). *The unitary intertwiners satisfy*

$$A_{L_1, L_0} A_{L_2, L_1} = \Gamma_\psi(L(L_0, L_1, L_2)) A_{L_2, L_0}.$$

Proof. First suppose the three Lagrangians are pairwise transverse. Write $W = L_0 \oplus L_1$ and $L_2 = \{by + y : y \in L_1\}$, where $b : L_1 \rightarrow L_0$ is invertible and $\langle by, y' \rangle$ is symmetric. Then the Leray form is

$$q_b(y) = -\frac{1}{2}\langle by, y \rangle.$$

For $f \in \mathcal{H}_{L_2}$, set $\phi(a) = f((a, 0))$, $a \in L_0$. The Heisenberg multiplication and left $H(L_2)$ -covariance give

$$f((y, 0)(a, 0)) = \psi(q_b(y))\phi(a - by).$$

For example, this follows by factoring the same group element as

$$(y, 0)(a, 0) = (by + y, 0)(a - by, -\frac{1}{2}\langle by, y \rangle).$$

Thus the unnormalized composite at the identity is

$$\int_{L_0} \int_{L_1} \psi(q_b(y))\phi(a - by) dy da.$$

Its value, interpreted through the Gaussian Fourier identity, is a positive real multiple of

$$\Gamma_\psi(q_b) \int_{L_0} \phi(a) da.$$

One may obtain this identity by Gaussian regularization in the Archimedean case or by integration over sufficiently large compact open subgroups in the non-Archimedean case. This avoids exchanging two integrals without a convergence justification. Since the two sides of the claimed operator identity are intertwiners, their ratio is a scalar; since their normalizations are unitary, its modulus is one. The positive measure factors therefore leave exactly $\Gamma_\psi(q_b)$.

For an arbitrary triple, the general composition theorem is obtained by symplectic reduction along the intersections of the Lagrangians. The remaining nondegenerate quadratic space is $K/\text{rad}(q_K)$; the elementary repeated-Lagrangian and hyperbolic factors contribute index 1. Applying the transverse calculation on the reduced factors proves the stated formula. This reduction is the general Lagrangian-intertwiner composition theorem; see [Kud, Chapter I, Section 4] and [Rao, Sections 3–4]. $\square$

Theorem 3.9. *With the preceding conventions,*

$$c_Y(g_1, g_2) = \Gamma_\psi(L(Y, g_1 Y, g_1 g_2 Y)).$$

Proof. Transport naturality and multiplicativity give

$$\begin{aligned} \Omega_Y(g_1)\Omega_Y(g_2) &= A_{g_1 Y, Y} A_{g_1 g_2 Y, g_1 Y} U_{g_1 g_2, Y} \\ &= \Gamma_\psi(L(Y, g_1 Y, g_1 g_2 Y)) A_{g_1 g_2 Y, Y} U_{g_1 g_2, Y}. \end{aligned}$$

Transfer this equality by Φ_Y and use the definition of c_Y . $\square$

If $q = L(Y, g_1 Y, g_1 g_2 Y)$ has dimension d and diagonal coefficients $a_1, \dots, a_d$, then

$$c_Y(g_1, g_2) = \Gamma_\psi(x^2)^d \gamma_\psi(\det q) \prod_{i < j} (a_i, a_j)_k.$$

When two Lagrangians in the triple coincide, the Leray form is zero in the Witt group and the cocycle is 1, in agreement with Proposition 3.5. The operator product and the displayed triple use the same order; replacing a right action by a left action without changing the triple would change this formula.

3.5. The metaplectic group as a twofold cover. Keep the polarization $W = X \oplus Y$ and the ordered symplectic basis

$$e = (e_1, \dots, e_n, f_1, \dots, f_n), \quad J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Vectors are columns, and throughout this subsection

$$P = P(Y) = \left\{ \begin{pmatrix} a & 0 \\ c & a^{-t} \end{pmatrix} \in \mathrm{Sp}(W) \right\}.$$

In particular, P is the *lower* Siegel parabolic. Write

$$m(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-t} \end{pmatrix}, \quad n(c) = \begin{pmatrix} I & 0 \\ c & I \end{pmatrix}, \quad c = c^t, \quad w = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Let w_j act as w on the last j hyperbolic planes and as the identity on the others. Thus $w_0 = 1$ and $w_n = w$. The Bruhat decomposition is

$$\mathrm{Sp}(W) = \coprod_{j=0}^n \Omega_j, \quad \Omega_j = P w_j P.$$

For $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, one has

$$j(g) = \mathrm{rank} \beta = n - \dim(Y \cap gY).$$

The relevant block is β because P stabilizes Y .

3.5.1. The open Bruhat cell.

Proposition 3.10. *For $g \in \Omega_n$, the matrices $\delta\beta^{-1}$ and $\beta^{-1}\alpha$ are symmetric, and*

$$g = n(\delta\beta^{-1}) m(\beta) w n(\beta^{-1}\alpha).$$

This factorization is unique in the indicated form.

Proof. The identities $g^t J g = J$ and $g J g^t = J$ imply $\beta^t \delta = \delta^t \beta$ and $\alpha \beta^t = \beta \alpha^t$, giving the two symmetry assertions. For symmetric c_1, c_2 , direct multiplication gives

$$n(c_1) m(b) w n(c_2) = \begin{pmatrix} b c_2 & b \\ c_1 b c_2 - b^{-t} & c_1 b \end{pmatrix}.$$

The upper-right, upper-left, and lower-right blocks force $b = \beta$, $c_2 = \beta^{-1}\alpha$, and $c_1 = \delta\beta^{-1}$. The remaining symplectic identity gives

$$\gamma = \delta\beta^{-1}\alpha - \beta^{-t},$$

which verifies the lower-left block. □

Corollary 3.11. *The multiplication map*

$$\mathrm{Sym}_n(k) \times \mathrm{GL}_n(k) \times \mathrm{Sym}_n(k) \longrightarrow \Omega_n, \quad (c_1, b, c_2) \longmapsto n(c_1) m(b) w n(c_2),$$

is an isomorphism of k -varieties. The cell $\Omega_n = D(\det \beta)$ is Zariski open and dense in $\mathrm{Sp}(W)$. For $n > 0$, every irreducible component of its complement has codimension one.

Proof. The inverse map is $g \mapsto (\delta\beta^{-1}, \beta, \beta^{-1}\alpha)$, which is regular on $D(\det \beta)$. The symplectic group is geometrically irreducible, and $\det \beta$ is nonzero on w . Its zero set is therefore a proper principal hypersurface. Its nonempty irreducible components have codimension one. □

In the Schrödinger model on $\mathcal{S}(X)$, take the self-dual measure for ψ and the positive Fourier transform. The standard unitary intertwiners on these elements are

$$\begin{aligned} r(m(a))\phi(z) &= |\det a|_k^{-1/2} \phi(a^{-1}z), \\ r(n(c))\phi(z) &= \psi\left(-\frac{1}{2}z^t c z\right) \phi(z), \\ r(w)\phi(z) &= \int_X \psi(z^t t) \phi(t) dt. \end{aligned}$$

The standard Bruhat normalization satisfies $r(p_1 g p_2) = r(p_1) r(g) r(p_2)$ for $p_1, p_2 \in P$. Consequently, on the open cell,

$$r(g)\phi(z) = |\det \beta|_k^{-1/2} \int_X \psi\left(-\frac{1}{2}z^t \delta \beta^{-1} z + t^t \beta^{-1} z - \frac{1}{2}t^t \beta^{-1} \alpha t\right) \phi(t) dt.$$

This formula follows directly from the factorization above; in particular its signs agree with the lower-parabolic convention.

3.5.2. Rao's construction. There is a well-defined square-class function

$$x_e : \mathrm{Sp}(W) \longrightarrow k^\times / k^{\times 2}, \quad x_e(p_1 w_j^{-1} p_2) = \det((p_1 p_2)|_Y) \pmod{k^{\times 2}}.$$

Its well-definedness is the Bruhat determinant calculation in [Rao, Lemma 5.1], with the indicated choice of Weyl representatives. It satisfies

$$x_e(p_1 g p_2) = x_e(p_1) x_e(g) x_e(p_2), \quad x_e(n(c) m(a)) = \det a \pmod{k^{\times 2}}.$$

The inverse in the first definition matters. With the present w_j ,

$$x_e(w_j) = (-1)^j, \quad x_e(g) = (-1)^n \det \beta \pmod{k^{\times 2}} \quad (g \in \Omega_n).$$

Indeed, w_j differs from w_j^{-1} by $m(a_j)$, where a_j is -1 on the exchanged coordinates and 1 on the others. The open-cell factorization then gives the second formula. If the definition uses w_j in place of w_j^{-1} , its value changes by $(-1)^{j(g)}$, and the corresponding Weil-index normalization must change with it.

Write

$$r(g_1) r(g_2) = c_{Y,\psi}(g_1, g_2) r(g_1 g_2), \quad c_{Y,\psi}(g_1, g_2) \in \mathbb{S}^1.$$

This fixes the direction of the cocycle convention.

Theorem 3.12 (Ranga Rao). *There is a normalized μ_2 -valued cocycle $c_{R,e}$ and a function $b_{e,\psi} : \mathrm{Sp}(W) \rightarrow \mathbb{S}^1$, with $b_{e,\psi}(1) = 1$, such that*

$$c_{R,e}(g_1, g_2) = \frac{b_{e,\psi}(g_1) b_{e,\psi}(g_2)}{b_{e,\psi}(g_1 g_2)} c_{Y,\psi}(g_1, g_2).$$

The cohomology class of this double-cover cocycle is independent of the nontrivial additive character ψ . It can be represented by Rao's cocycle, expressed in terms of $j(g)$, $x_e(g)$, Hilbert symbols, and the quadratic space of the Leray invariant. The resulting extension carries its standard locally compact covering-group topology, with a continuous embedding into the unitary scalar extension. For the usual Rao representative, its restriction to the Siegel parabolic is

$$c_{R,e}(p_1, p_2) = (\det(p_1|_Y), \det(p_2|_Y))_k.$$

The reduction to μ_2 is the substantive normalization theorem; see [Rao, Theorem 5.3 and Corollary 5.5] or [Kud, Chapter I]. Here $b_{e,\psi}$ denotes the cochain after transporting that theorem to the column-vector and Fourier conventions above. This formulation also fixes its reciprocal: the corrected operators are $b_{e,\psi}(g) r(g)$.

Definition 3.13. Define $\mathrm{Mp}^{(2)}(W)_e$ as the set $\mathrm{Sp}(W) \times \mu_2$, with multiplication

$$(g_1, \epsilon_1)(g_2, \epsilon_2) = (g_1 g_2, \epsilon_1 \epsilon_2 c_{R,e}(g_1, g_2)).$$

It is the *metaplectic double cover*. Its scalar pushouts are

$$\mathrm{Mp}^{\mathbb{S}^1}(W)_e = \mathrm{Mp}^{(2)}(W)_e \times_{\mu_2} \mathbb{S}^1, \quad \mathrm{Mp}^{\mathbb{C}^\times}(W)_e = \mathrm{Mp}^{(2)}(W)_e \times_{\mu_2} \mathbb{C}^\times.$$

In these coordinates, the isomorphism to the unitary intertwiner extension is

$$(g, z) \mapsto (g, z b_{e,\psi}(g) r(g)),$$

and the Weil representation is

$$\omega_\psi(g, z) = z b_{e,\psi}(g) r(g).$$

The same formula gives the isomorphism for the complex-scalar extension. The topology is the metaplectic topology furnished by the theorem; a Bruhat-coordinate section need not be globally continuous. In particular, a product description of the underlying set does not assert that the topology is a product topology.

Lemma 3.14. *An automorphism of a scalar metaplectic extension which induces the identity both on its scalar kernel and on $\mathrm{Sp}(W)$ is the identity.*

Proof. Such an automorphism differs from the identity by a character of $\mathrm{Sp}(W)$. For a nondiscrete local field of characteristic different from 2, this group is perfect when $n > 0$, so that character is trivial. For $n = 0$ the assertion is immediate. $\square$

Corollary 3.15. *The metaplectic scalar extensions obtained from different polarizations, symplectic bases, and nontrivial additive characters are canonically isomorphic as central extensions with fixed kernel and quotient. The same is true of the double covers.*

Proof. Rao's theorem gives the isomorphisms of scalar extensions, and Lemma 3.14 gives their uniqueness. In the coordinates defined by $c_{R,e}$, the character

$$\lambda_e(g, z) = z^2$$

is a homomorphism, because $c_{R,e}^2 = 1$. Its kernel is the double cover. This character is uniquely determined by its restriction $z \mapsto z^2$ to the scalar center: the quotient of any two such characters factors through $\mathrm{Sp}(W)$. Hence the canonical scalar-extension isomorphisms carry the double covers to each other. $\square$

Remark 3.16. Independence of the central extension from ψ does not mean independence of the Weil representation from ψ . The latter retains its additive-character parameter. The scalar extensions also have different kernels from the double cover; these three groups should be distinguished when discussing a splitting.

3.6. Splitting and nonsplitting of the metaplectic extension.

Proposition 3.17. *The unitary-scalar and complex-scalar metaplectic extensions split over $P(Y)$. The splitting afforded by $p \mapsto r(p)$ has a restriction to*

$$P(Y)^1 = \{p \in P(Y) : \det(p|_Y) = 1\}$$

which is independent of the choice of splitting over $P(Y)$.

Proof. The formulas for $r(n(c))$ and $r(m(a))$ give genuine representations of N and M . They satisfy

$$m(a)n(c)m(a)^{-1} = n(a^{-t}ca^{-1}), \quad r(m(a))r(n(c))r(m(a))^{-1} = r(n(a^{-t}ca^{-1})).$$

Thus $r|_P$ is a representation and $c_{Y,\psi}|_{P \times P} = 1$. Two splittings differ by a character of P . Every such character kills N : conjugation by $m(tI)$ sends $n(c)$ to $n(t^{-2}c)$, and one can choose $t^2 \neq 1$. On $M \simeq \mathrm{GL}_n(k)$ a character factors through the determinant, since the commutator subgroup is $\mathrm{SL}_n(k)$ for $n \geq 2$. For $n = 1$ this factorization is immediate. The two splittings therefore agree on $P(Y)^1$. $\square$

Lemma 3.18. *The lift $n \mapsto (n, r(n))$ is the unique splitting of $N(Y)$ in the scalar metaplectic extension which is normalized by $P(Y)$. It is also the restriction of the canonical splitting of the double cover over the unipotent group $N(Y)$.*

Proof. For two such scalar splittings, their quotient is a character χ of $N(Y)$ invariant under conjugation by M . In particular $\chi(n(t^{-2}c)) = \chi(n(c))$ for every c . Choosing $t^2 \neq 1$ shows $\chi(n((t^{-2} - 1)c)) = 1$ for every c , hence $\chi = 1$. Compose this normalized scalar splitting with the character λ whose kernel is the double cover. The resulting character of N is again invariant under M , hence trivial by the same argument. The splitting therefore takes values in the double cover. $\square$

Remark 3.19. The splitting of the scalar extension over P need not take values in its double-cover subgroup. For example, let $k = \mathbb{Q}_2$ and $n = 1$. For $h = m(-1)$, the Rao restriction formula gives

$$c_{R,e}(h, h) = (-1, -1)_{\mathbb{Q}_2} = -1.$$

Every lift of the order-two element h to the double cover has square equal to the nonidentity central element. Thus the double cover does not split over P in this example, although both scalar extensions do.

3.6.1. The rank-one obstruction. For this calculation, put

$$q_a(t) = \frac{1}{2}at^2, \quad G_\psi(a) = \Gamma_\psi(q_a), \quad g_\psi(a) = G_\psi(a)/G_\psi(1), \quad a \in k^\times,$$

where Γ_ψ is the Weil index. The standard identities are

$$G_\psi(-a) = G_\psi(a)^{-1}, \quad g_\psi(a)g_\psi(b) = (a, b)_k g_\psi(ab).$$

The second identity says precisely that the normalized one-dimensional Weil index is a quadratic refinement of the Hilbert symbol, rather than in general a multiplicative character.

Lemma 3.20. *In dimension two, with $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $n(b) = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}$, one has*

$$c_{Y,\psi}(w, n(b)w) = G_\psi(-b), \quad b \in k^\times.$$

Proof. The matrix $wn(b)w$ equals $\begin{pmatrix} -1 & b \\ 0 & -1 \end{pmatrix}$, so the open-cell formula gives

$$r(wn(b)w)\phi(z) = |b|_k^{-1/2} \int_k \psi\left(\frac{(z+t)^2}{2b}\right) \phi(t) dt.$$

On the other hand, $r(w)r(n(b)w)$ has the inner Gaussian integral

$$\int_k \psi\left(-\frac{b}{2}s^2 + (z+t)s\right) ds = G_\psi(-b)|b|_k^{-1/2} \psi\left(\frac{(z+t)^2}{2b}\right).$$

The integral is interpreted as a tempered distribution, or by the usual oscillatory limit. The Gaussian Fourier identity proves the formula without an unjustified interchange of nonabsolutely convergent integrals. $\square$

Theorem 3.21. *Suppose $\dim W > 0$. If $k = \mathbb{R}$ or k is nonarchimedean of characteristic different from 2, neither the scalar metaplectic extension nor its double cover splits over $\mathrm{Sp}(W)$. For $k = \mathbb{C}$, both extensions split.*

Proof. First let $\dim W = 2$, so $\mathrm{Sp}(W) = \mathrm{SL}_2(k)$. If the scalar extension split, there would be a function $\alpha : \mathrm{SL}_2(k) \rightarrow \mathbb{S}^1$ with

$$c_{Y,\psi}(g_1, g_2) = \alpha(g_1)\alpha(g_2)\alpha(g_1g_2)^{-1}.$$

For the complex-scalar extension one can initially take α valued in $\mathbb{C}^\times$; its absolute value is a character of the perfect group $\mathrm{SL}_2(k)$, hence is trivial. Since r is P -biequivariant, the cocycle is 1 whenever one entry belongs to P . It follows that

$$\alpha(p_1gp_2) = \alpha(p_1)\alpha(g)\alpha(p_2).$$

The character $\alpha|_P$ kills its unipotent radical, so $\alpha(m(a)) = \eta(a)$ for a character $\eta : k^\times \rightarrow \mathbb{S}^1$. Set $A = \alpha(w)$. On the open cell the factorization above gives $\alpha(g) = A\eta(\beta)$.

Apply the coboundary identity to $(w, n(b)w)$. Its numerator is A^2 , and its denominator is $A\eta(b)$. By Lemma 3.20,

$$G_\psi(-b) = A\eta(b)^{-1}.$$

Taking $b = 1$ gives $A = G_\psi(-1)$, so

$$\eta(b) = \frac{G_\psi(b)}{G_\psi(1)} = g_\psi(b).$$

Multiplicativity of η would force $(a, b)_k = 1$ for all $a, b \in k^\times$. This contradicts the nontriviality of the local Hilbert symbol over $\mathbb{R}$ and nonarchimedean local fields.

For general W , choose a symplectic plane W_1 and an orthogonal decomposition $W = W_1 \oplus W_1^\perp$. The restriction of the scalar extension to $\mathrm{Sp}(W_1)$ is the rank-one extension: in the tensor

Schrödinger model its intertwiners are $r_{W_1}(g) \otimes 1$. A splitting would restrict to a splitting of that extension, which is impossible. A splitting of the double cover would also split its scalar pushout.

Over $\mathbb{C}$, quadratic Hilbert symbols and the relevant Weil indices are trivial. Rao's normalized cocycle is identically 1 [Rao, Corollary 5.7], so the normalized intertwiners give a representation of $\mathrm{Sp}(W)$ and both extensions split. $\square$

4. THETA CORRESPONDENCE

In this section F is a finite extension of $\mathbb{Q}_p$, and all representations are complex and smooth. We retain a nontrivial unitary character ψ of F . We outline the local theory following [Kud, Chapters II–IV], concentrating on orthogonal–symplectic pairs. The statements about Howe duality are given in their proved form. For a representation π , $\pi^\vee$ denotes its smooth contragredient.

4.1. Reductive dual pairs.

Definition 4.1. A *reductive dual pair* in a symplectic group $\mathrm{Sp}(\mathbb{W})$ is a pair of reductive subgroups (G, G') which are mutual centralizers:

$$Z_{\mathrm{Sp}(\mathbb{W})}(G) = G', \quad Z_{\mathrm{Sp}(\mathbb{W})}(G') = G.$$

Let (V, B) be a nondegenerate symmetric space of dimension m , with $q_V(v) = \frac{1}{2}B(v, v)$, and let W be symplectic of dimension $2n$. On $\mathbb{W} = V \otimes_F W$ put

$$\langle v \otimes w, v' \otimes w' \rangle_{\mathbb{W}} = B(v, v') \langle w, w' \rangle_W.$$

The form is alternating and nondegenerate. The commuting actions give

$$\mathrm{O}(V) \times \mathrm{Sp}(W) \longrightarrow \mathrm{Sp}(\mathbb{W}), \quad (h, g) \longmapsto h \otimes g.$$

The individual images form a reductive dual pair. When both spaces have positive dimension, the displayed homomorphism has central kernel $\{(1, 1), (-1, -1)\}$; it is not an embedding of the product. The centralizer assertion follows from the standard tensor-space centralizer calculation for the full orthogonal and symplectic groups.

Other examples include unitary pairs associated with Hermitian and skew-Hermitian spaces, and the type II pair $\mathrm{GL}(A) \times \mathrm{GL}(B)$ on $E \oplus E^*$, where $E = \mathrm{Hom}(B, A)$ has its usual evaluation pairing with E^* . On E , the action is $(a, b)T = aTb^{-1}$, and on E^* it is the contragredient action. The orthogonal–symplectic example already exhibits the main features of the local correspondence.

The Weil representation is initially attached to the metaplectic extension of $\mathrm{Sp}(\mathbb{W})$. Its restrictions require the scalar normalizations described in [Kud, Sections II.3–II.5]. After these normalizations it gives a representation $\omega_{V, W, \psi}$ of

$$\mathrm{O}(V) \times G(W), \quad G(W) = \begin{cases} \mathrm{Sp}(W), & m \text{ even,} \\ \mathrm{Mp}^{(2)}(W), & m \text{ odd.} \end{cases}$$

In the odd-dimensional case the representation on the second factor is genuine. The parity is explained by the tensor embedding: the central -1 of the source metaplectic cover acts after pull-back by $(-1)^m$. Thus the symplectic action descends precisely when m is even. These normalized product actions should not be identified with an arbitrary choice of lifts in the ambient double cover.

4.2. The orthogonal–symplectic Schrödinger model. Choose $W = X \oplus Y$ as before. The oscillator space is

$$\mathcal{S}(V \otimes X) \simeq \mathcal{S}(V^n).$$

We regard $x = (x_1, \dots, x_n)$ as a row of vectors in V , while the matrices on W still act on column vectors. Put

$$\mathrm{Gr}(x) = (B(x_i, x_j))_{i,j}, \quad \mathcal{F}_V \phi(x) = \int_{V^n} \phi(y) \psi \left(\sum_i B(x_i, y_i) \right) dy,$$

with the self-dual measure for $\psi \circ B$.

For any m , define the signed discriminant and its quadratic character by

$$d(V) = (-1)^{m(m-1)/2} \det B \pmod{F^{\times 2}}, \quad \chi_V(t) = (t, d(V))_F.$$

Suppose now that m is even, so $d(V) = (-1)^{m/2} \det B$. With the lower Siegel parabolic and positive Fourier convention of Section 3, the formulas are

$$\begin{aligned} (\omega(h)\phi)(x) &= \phi(h^{-1}x), & h &\in O(V), \\ (\omega(m(a))\phi)(x) &= \chi_V(\det a) |\det a|_F^{-m/2} \phi(xa^{-t}), \\ (\omega(n(c))\phi)(x) &= \psi\left(-\frac{1}{2} \operatorname{tr}(\operatorname{Gr}(x)c)\right) \phi(x), & c &= c^t, \\ \omega(w)\phi &= \Gamma_\psi(q_V)^n \mathcal{F}_V \phi. \end{aligned}$$

Here h^{-1} acts on every component of x . The determinant factor in the second formula is the square root of the Jacobian of the linear change on V^n . The character χ_V and the Weil index in the last formula correct the projective action to a representation. These elements generate $\operatorname{Sp}(W)$.

To check the Fourier scalar, take $n = 1$. The Gaussian relation for q_V and the identity

$$\Gamma_\psi(aq_V) = \chi_V(a) \Gamma_\psi(q_V) \quad (m \text{ even})$$

give the rank-one symplectic relations. In particular, $\Gamma_\psi(q_V)^2 = \chi_V(-1)$, so $\omega(w)^2$ agrees with $\omega(m(-1))$. The sign of the quadratic multiplier and the positive Fourier transform here are the counterparts of the opposite polarization convention used in [Kud, Section II.4]. For odd m , the same construction uses the genuine factors on the metaplectic Levi; the cover cannot simply be suppressed.

4.3. Big theta, small theta, and Howe duality. Write $\omega = \omega_{V,W,\psi}$ and let (G, G') be either ordering of its two factors. If $G = \operatorname{Mp}^{(2)}(W)$, only genuine irreducible representations of G can occur.

Definition 4.2. For an irreducible admissible G -representation π , put

$$N(\pi) = \bigcap_{f \in \operatorname{Hom}_G(\omega, \pi)} \ker f.$$

The maximal π -isotypic quotient has the form

$$\omega/N(\pi) \simeq \pi \boxtimes \Theta_{G \rightarrow G'}(\pi).$$

The smooth G' -representation $\Theta_{G \rightarrow G'}(\pi)$ is the *big theta lift*. If no such quotient occurs, it is zero. Equivalently,

$$\Theta_{G \rightarrow G'}(\pi) \simeq (\omega \otimes \pi^\vee)_G,$$

where the subscript denotes coinvariants for the diagonal G -action.

The two group actions commute, so $N(\pi)$ is stable under G' . The quotient and coinvariant descriptions are the standard isotypic quotient construction; see [Kud, Section II.2]. For every smooth representation τ of G' , they give the universal property

$$\operatorname{Hom}_{G'}(\Theta_{G \rightarrow G'}(\pi), \tau) \simeq \operatorname{Hom}_{G \times G'}(\omega, \pi \boxtimes \tau).$$

In this setting one uses quotients, rather than assuming that the oscillator representation is a direct sum of irreducibles.

Theorem 4.3 (Finiteness and Howe duality). *For the orthogonal-symplectic pair, $\Theta(\pi)$ has finite length. Its maximal semisimple quotient, denoted $\theta(\pi)$ and called the small theta lift, is zero or irreducible. Moreover,*

$$\theta(\pi) \simeq \theta(\pi') \neq 0 \implies \pi \simeq \pi'.$$

The same assertions hold with the two members interchanged.

The finiteness statement is due to Kudla. The Howe duality theorem in arbitrary residual characteristic is proved in [GT, Theorem 1.2]. Thus the correspondence is a bijection between the subsets of irreducible representations having nonzero lifts for a fixed pair and fixed ψ ; it need not be defined on every irreducible representation of either group.

Corollary 4.4. *For irreducible admissible π and τ of the two members,*

$$\dim \operatorname{Hom}_{G \times G'}(\omega, \pi \boxtimes \tau) \leq 1.$$

It is nonzero precisely when $\tau \simeq \theta(\pi)$.

Proof. Every map from the finite-length $\Theta(\pi)$ to the irreducible τ factors through its maximal semisimple quotient. Apply Theorem 4.3, the universal property, and Schur's lemma. $\square$

For general input, $\Theta(\pi)$ can be reducible and contains more information than its irreducible quotient $\theta(\pi)$. If π is supercuspidal, however, its big theta lift is zero or irreducible; thus $\Theta(\pi) = \theta(\pi)$ whenever it is nonzero [GT, Theorem 1.1(i)].

4.4. Witt towers and first occurrence. Let $\mathbb{H}$ be a hyperbolic quadratic plane and fix an anisotropic quadratic space V_{an} . Its *Witt tower* is

$$\mathcal{V} = \{V_r = V_{\text{an}} \oplus \mathbb{H}^{\oplus r} : r \geq 0\}.$$

Here r is the Witt index, while $\dim V_r = \dim V_{\text{an}} + 2r$. The parity and the character χ_{V_r} are constant in the tower, so the symplectic member $G(W)$ and its cover are fixed as r varies. Choose the associated normalizations compatibly along the tower.

Theorem 4.5 (Persistence and eventual nonvanishing). *Fix $\pi \in \operatorname{Irr}(G(W))$, genuine when required. If $\Theta_{V_r}(\pi) \neq 0$, then $\Theta_{V_{r+1}}(\pi) \neq 0$. Every such π occurs eventually; a sufficient bound is $r \geq 2n$ when $\dim W = 2n$.*

These are the persistence and stable-range results of [Kud, Section III.4, Propositions 4.1 and 4.3]. The displayed bound concerns the Witt index r , not the dimension of V_r .

Definition 4.6. The *first occurrence index* in this tower is

$$r_{\mathcal{V}}(\pi) = \min\{r \geq 0 : \Theta_{V_r}(\pi) \neq 0\}.$$

It is finite, and persistence says that the nonvanishing set is exactly $\{r \geq r_{\mathcal{V}}(\pi)\}$. First occurrence depends on the chosen Witt tower; specifying only the dimension does not specify the target quadratic space.

One can instead fix $\tau \in \operatorname{Irr}(\operatorname{O}(V))$ and vary symplectic spaces W_n of dimension $2n$. Define

$$n(\tau) = \min\{n \geq 0 : \Theta_{W_n}(\tau) \neq 0\}.$$

Persistence holds in this direction as well, and $n \geq \dim V$ is a sufficient nonvanishing bound [Kud, Proposition III.4.5]. Rank zero is understood as the formal bottom of the tower, with Schwartz space $\mathbb{C}$ and the trivial symplectic action.

Theorem 4.7 (Supercuspidals at first occurrence). *If the input is supercuspidal, its first nonzero theta lift in a fixed tower is supercuspidal. At every later level, its nonzero lift is irreducible but is not supercuspidal.*

This is [Kud, Theorems III.6.1–III.6.2]. The distinction is between the first occurrence and later occurrences in that same tower. Later lifts are controlled by parabolic induction from the first one, with characters on the additional general linear factors.

Example 4.8 (The two characters of $\operatorname{O}(1)$). Let V be one-dimensional. Then $\operatorname{O}(V) = \{\pm 1\}$, and its action on $\mathcal{S}(F^n)$ is $\phi(x) \mapsto \phi(-x)$. For $n \geq 1$ the two lifts are

$$\Theta_{W_n}(\mathbf{1}) = \mathcal{S}(F^n)^+, \quad \Theta_{W_n}(\operatorname{sgn}) = \mathcal{S}(F^n)^-,$$

the even and odd parts of the genuine Weil representation of $\operatorname{Mp}^{(2)}(W_n)$. Both are nonzero and irreducible. At $n = 0$ only the trivial character occurs. Consequently

$$n(\mathbf{1}) = 0, \quad n(\operatorname{sgn}) = 1.$$

This example realizes the quotient definition directly, since the source group is finite and its two isotypic projections are $\frac{1}{2}(1 \pm \omega(-1))$.

4.5. Jacquet modules and degenerate principal series. Kudla's proofs use Jacquet modules of the Weil representation. For example, let $Q_t \subset \mathrm{O}(V_r)$ stabilize an isotropic t -plane. Its Levi factor is $\mathrm{GL}_t(F) \times \mathrm{O}(V_{r-t})$. In a mixed Schrödinger model, partial Fourier transform reduces the calculation of its Jacquet module to a filtration by the ranks of linear maps between the relevant isotropic spaces. The rank-zero term contains the Weil representation for (V_{r-t}, W) , with the appropriate Levi character; the other terms are induced from smaller dual pairs. The analogous calculation for a parabolic in $\mathrm{Sp}(W)$ reduces the symplectic rank. This is the Jacquet-module filtration in [Kud, Section III.8, Theorems 8.1–8.2].

The rank-zero piece explains persistence: a nonzero quotient at a smaller level yields one at the next level. For supercuspidal input, vanishing of proper Jacquet modules eliminates the possible earlier boundary contributions. This is the mechanism behind the first-occurrence theorem above; the filtration also determines the parabolic induction data of later lifts.

There is a concrete link with degenerate principal series. Assume m is even, and retain $P = P(Y) \subset \mathrm{Sp}(W)$. For $\phi \in \mathcal{S}(V^n)$, put

$$f_\phi(g) = (\omega(g)\phi)(0).$$

The generator formulas give

$$f_\phi(n(c)m(a)g) = \chi_V(\det a)|\det a|^{-m/2}f_\phi(g).$$

Thus evaluation at the origin defines a nonzero equivariant map

$$\mathcal{S}(V^n) \longrightarrow \mathrm{Ind}_P^{\mathrm{Sp}(W)}(\chi_V \circ \det | \det |^{-m/2}),$$

where this induction is unnormalized. For the lower Siegel parabolic, $\delta_P(m(a)) = |\det a|^{-(n+1)}$. In normalized induction the parameter is therefore

$$s_0 = \frac{n+1-m}{2}, \quad i_P^{\mathrm{Sp}(W)}(\chi_V \circ \det | \det |^{s_0}).$$

Because evaluation at 0 is $\mathrm{O}(V)$ -invariant, the map factors through $\Theta_{\mathrm{O}(V) \rightarrow \mathrm{Sp}(W)}(\mathbf{1})$. Its image is an explicit theta quotient inside a degenerate principal series. This is the construction studied in [Kud, Section III.5], expressed in our lower-parabolic convention.

4.6. Seesaw dual pairs. Suppose (G, G') and (H, H') are dual pairs in the same ambient symplectic space, with

$$H \subset G, \quad G' \subset H'.$$

Use compatible covering groups and normalizations, so that both correspondences are obtained from the same oscillator representation ω . This arrangement is called a *seesaw*.

Proposition 4.9 (Seesaw identity). *For irreducible admissible σ of H and τ of G' , one has*

$$\begin{aligned} \mathrm{Hom}_H(\Theta_{G' \rightarrow G}(\tau)|_H, \sigma) &\simeq \mathrm{Hom}_{H \times G'}(\omega, \sigma \boxtimes \tau) \\ &\simeq \mathrm{Hom}_{G'}(\Theta_{H \rightarrow H'}(\sigma)|_{G'}, \tau). \end{aligned}$$

Proof. Factor an $H \times G'$ -equivariant map from ω first through the maximal τ -isotypic quotient for G' . This gives the first space. Factoring first through the maximal σ -isotypic quotient for H gives the last space. Both factorizations use the universal property of Definition 4.2. $\square$

For example, if $V = V_1 \perp V_2$, then $\mathrm{O}(V_1) \times \mathrm{O}(V_2) \subset \mathrm{O}(V)$, while $\mathrm{Sp}(W)$ embeds diagonally in $\mathrm{Sp}(W) \times \mathrm{Sp}(W)$. The tensor identity

$$\omega_{V,W} \simeq \omega_{V_1,W} \otimes \omega_{V_2,W}$$

with compatible actions is the corresponding seesaw construction. One can therefore convert a branching problem on an orthogonal group into a question about theta lifts on the symplectic side. The identity uses big theta lifts; replacing them by their small quotients is not justified without further hypotheses. See [Kud, Chapter IV, Section 1] for this method and the doubling seesaw.

APPENDICES

5. A RESULT IN p -ADIC ANALYSIS

We record the passage between the Schwartz model and its Hilbert space completion. Haar measures are chosen as in the preceding sections.

Theorem 5.1 (Extension from a dense subspace). *Let E be a normed space, $D \subset E$ a dense linear subspace, and F a Banach space. Every bounded linear map $T : D \rightarrow F$ extends uniquely to a bounded linear map $\tilde{T} : E \rightarrow F$, with $\|\tilde{T}\| = \|T\|$. If $E = F = \mathcal{H}$ is a Hilbert space and T is an isometry, then $\tilde{T}$ is an isometry. It is unitary if $T(D)$ is dense in $\mathcal{H}$.*

Proof. For $v \in E$, choose $v_j \in D$ with $v_j \rightarrow v$ and set $\tilde{T}v = \lim_j T v_j$. The bound

$$\|T v_i - T v_j\| \leq \|T\| \|v_i - v_j\|$$

shows that the limit exists and is independent of the approximating sequence. Linearity and $\|\tilde{T}\| \leq \|T\|$ follow by taking limits; restriction to D gives the reverse inequality. Two continuous extensions agree on the closure of D , proving uniqueness. If T is an isometry, the same limiting argument gives $\|\tilde{T}v\| = \|v\|$. Its range is closed, and is all of $\mathcal{H}$ when it contains the dense subspace $T(D)$. $\square$

The space $\mathcal{S}(X)$ is dense in $L^2(X)$. Thus an isometric automorphism of $\mathcal{S}(X)$ extends uniquely to a unitary operator on $L^2(X)$. An arbitrary unitary operator on $L^2(X)$ need not preserve $\mathcal{S}(X)$; preservation in the Weil representation requires an additional argument.

Proposition 5.2. *Suppose k is non-Archimedean. The smooth vectors of the unitary Schrödinger representation on $L^2(X)$ are precisely $\mathcal{S}(X)$, where functions are identified up to equality almost everywhere. Consequently, every unitary operator implementing a symplectic automorphism of the Heisenberg group preserves $\mathcal{S}(X)$.*

Proof. A smooth vector is fixed by translations by some lattice $A \subset X$ and by modulations from some lattice $B \subset Y$. Averaging over A gives an A -invariant, locally constant representative. Averaging the modulation action over B multiplies a function by the characteristic function of the compact annihilator

$$B^\perp = \{x \in X : \psi(\langle x, b \rangle) = 1 \text{ for every } b \in B\}.$$

The representative is therefore compactly supported, hence belongs to $\mathcal{S}(X)$. Conversely, a compactly supported locally constant function is fixed by sufficiently small translations, modulations, and central elements, so is smooth.

If $T\rho_\psi(h)T^{-1} = \rho_\psi(gh)$ for $g \in \text{Sp}(W)$, then T carries a vector fixed by a compact open subgroup K to a vector fixed by gK . Thus T and T^{-1} preserve the smooth vectors. $\square$

For Archimedean k , preservation of the Schwartz space follows from the explicit Fourier, linear-change-of-variable, and quadratic-multiplication operators. It does not follow merely from boundedness on L^2 .

Theorem 5.3 (Unitary Stone–von Neumann theorem). *Two irreducible strongly continuous unitary representations of $H(W)$ with the same nontrivial central character ψ are unitarily equivalent. For a polarization $W = X \oplus Y$, the Schrödinger representation is realized on $L^2(X)$, with translations from X and multiplication by the characters $x \mapsto \psi(\langle x, y \rangle)$ from Y .*

The operators constructed on $\mathcal{S}(X)$ above extend by Theorem 5.1 to this unitary model. In the non-Archimedean case, Proposition 5.2 also recovers the Schwartz model from the unitary one.

6. SOME SYMPLECTIC GEOMETRY

6.1. The Leray invariant. Let $\Omega(W)$ be the set of Lagrangian subspaces of W . Two Lagrangians L_0, L_1 are *transverse* if $L_0 \cap L_1 = 0$. Since both have dimension n , this is equivalent to $W = L_0 \oplus L_1$.

First consider a pairwise transverse ordered triple (L_0, L_1, L_2) . Relative to $W = L_0 \oplus L_1$, write

$$L_2 = \{\beta y + y : y \in L_1\}, \quad \beta : L_1 \xrightarrow{\sim} L_0.$$

Transversality to L_0 gives the graph description, and transversality to L_1 gives invertibility of β . Isotropy of L_2 says precisely that

$$b_\beta(y, y') = \langle \beta y, y' \rangle$$

is symmetric. It is nondegenerate because β is invertible. The element

$$u_\beta = \begin{pmatrix} I & \beta \\ 0 & I \end{pmatrix}$$

fixes L_0 pointwise and takes L_1 to L_2 . These properties determine it uniquely: the pointwise stabilizer of L_0 in the symplectic group is precisely this Siegel unipotent subgroup.

With the intertwiner conventions of these notes, the associated quadratic form is

$$q_\beta(y) = -\frac{1}{2} \langle \beta y, y \rangle.$$

The sign and the factor $1/2$ are part of the convention: the polar bilinear form of q_β is $-b_\beta$.

Here is a definition that also applies when the triple is not transverse.

Definition 6.1 (Leray–Maslov form). For $(L_0, L_1, L_2) \in \Omega(W)^3$, put

$$K = \{(x_0, x_1, x_2) \in L_0 \oplus L_1 \oplus L_2 : x_0 + x_1 + x_2 = 0\}, \quad q_K(x_0, x_1, x_2) = -\frac{1}{2} \langle x_0, x_1 \rangle.$$

Let B_K be its polar form and let $R_K = \text{rad}(B_K)$. The *Leray invariant* $L(L_0, L_1, L_2)$ is the isometry class of the nondegenerate quadratic space $(K/R_K, \bar{q}_K)$. Its Witt class is the Maslov index of the triple.

Since the characteristic is zero, $q_K(x) = \frac{1}{2} B_K(x, x)$, so it vanishes on R_K and descends to the quotient. For the transverse graph above, the map

$$y \mapsto (-\beta y, -y, \beta y + y)$$

identifies q_K with q_β , proving agreement with the first construction.

We also explain the symplectic reduction used to remove intersections.

Lemma 6.2. *If $M \subset W$ is isotropic, then $W_M = M^\perp/M$ is symplectic, with*

$$\langle w + M, w' + M \rangle_M = \langle w, w' \rangle.$$

For a Lagrangian $L \subset W$, its reduced image

$$L^M = ((L \cap M^\perp) + M)/M$$

is a Lagrangian in W_M .

Proof. The induced form is well-defined, and its radical is zero because $(M^\perp)^\perp = M$. The reduced image is isotropic. Moreover,

$$\dim(L \cap M^\perp) = n - \dim M + \dim(L \cap M),$$

so $\dim L^M = n - \dim M = \frac{1}{2} \dim W_M$. □

For a triple, take

$$M = (L_0 \cap L_1) + (L_1 \cap L_2) + (L_2 \cap L_0).$$

This is isotropic: any two summands lie together in one of the L_i . Its reduced Lagrangians are pairwise transverse. Indeed, if representatives $x_i \in L_i \cap M^\perp$ satisfy $x_0 - x_1 = a + b + c \in M$, with $a \in L_0 \cap L_1$, $b \in L_1 \cap L_2$, $c \in L_2 \cap L_0$, then

$$x_0 - c = x_1 + a + b \in L_0 \cap L_1,$$

so both reduced representatives are zero.

This reduction gives the same quadratic space as Definition 6.1. To see this, every component of a tuple in K is orthogonal to M . Reduction therefore maps K onto the kernel space of the reduced triple. Its kernel is

$$R_K = \{(a, -a, 0)\} + \{(0, b, -b)\} + \{(-c, 0, c)\},$$

where $a \in L_0 \cap L_1$, $b \in L_1 \cap L_2$, and $c \in L_2 \cap L_0$. These subspaces are in the radical by direct pairing, and the reduced graph form is nondegenerate; thus they give the whole radical. This also justifies the reduction for arbitrary triples, including coincident Lagrangians.

The construction is symplectically invariant. Cyclic permutation of the three Lagrangians preserves its isometry class; exchanging two of them negates the form. A triple with a repeated Lagrangian has zero Leray form. The four-term relation holds in the Witt group:

$$[L(L_0, L_1, L_2)] + [L(L_0, L_2, L_3)] = [L(L_0, L_1, L_3)] + [L(L_1, L_2, L_3)].$$

These are the geometric identities underlying the Weil-index cocycle; see [Kud, Chapter I, Section 3] and [Rao]. They are identities of Witt classes where indicated, rather than assertions that all displayed orthogonal sums are isometric.

7. CALCULATION OF THE WEIL INDEX

We specify the normalization before making computations. For a nondegenerate quadratic form $q(x) = \frac{1}{2}x^t Ax$ on k^r , use the self-dual measure for the Fourier transform

$$\widehat{f}(\xi) = \int_{k^r} f(x) \psi(x^t \xi) dx.$$

The Weil index $\Gamma_\psi(q)$ is characterized by the identity of tempered distributions

$$\widehat{\psi(q)}(\xi) = \Gamma_\psi(q) |\det A|_k^{-1/2} \psi\left(-\frac{1}{2} \xi^t A^{-1} \xi\right).$$

In particular, for $q(x) = ax^2$, the determinant factor is $|2a|_k^{-1/2}$. Write $\Gamma_\psi(0) = 1$ for the zero-dimensional form. The index is unchanged by isometry and is multiplicative under orthogonal sums. Also,

$$\Gamma_\psi(-q) = \Gamma_\psi(q)^{-1}, \quad \Gamma_\psi(H) = 1$$

for a hyperbolic plane H . Thus it depends only on the Witt class. These conventions match the quadratic phase in the preceding section; see [Wei, Kud].

7.1. $k = \mathbb{R}$. Write $\psi(t) = e^{2\pi i \lambda t}$, with $\lambda \neq 0$. The regularized Gaussian integral gives

$$\Gamma_\psi(ax^2) = \exp\left(\frac{\pi i}{4} \operatorname{sgn}(\lambda a)\right).$$

For example, insert a factor $e^{-\varepsilon x^2}$ in the oscillatory integral and let $\varepsilon \downarrow 0$; the square root of $\varepsilon - 2\pi i \lambda a$ has limiting argument $-\frac{\pi}{4} \operatorname{sgn}(\lambda a)$. If q has signature (p_+, p_-) , diagonalization and multiplicativity give

$$\Gamma_\psi(q) = \exp\left(\frac{\pi i}{4} \operatorname{sgn}(\lambda)(p_+ - p_-)\right).$$

7.2. $k = \mathbb{C}$. Every nontrivial unitary additive character has the form $\psi(z) = e^{2\pi i \operatorname{Re}(cz)}$ for $c \neq 0$. A nonzero complex one-dimensional form az^2 , viewed through $\operatorname{Re}(caz^2)$, has real signature $(1, 1)$. Hence its index is 1. Every nondegenerate complex form diagonalizes, so

$$\Gamma_\psi(q) = 1$$

for all such forms.

7.3. k is **non-Archimedean**. Let q be nondegenerate on a finite-dimensional space V , with polar form B_q . Choose a lattice L sufficiently small that $\psi(q(l)) = 1$ for every $l \in L$, and put

$$L^\# = \{x \in V : \psi(B_q(x, l)) = 1 \text{ for all } l \in L\}.$$

Then $L \subset L^\#$; the inclusion is in this direction. The function $\psi(q)$ descends to $L^\#/L$, because $q(x+l) = q(x) + q(l) + B_q(x, l)$. Orthogonality of characters gives

$$(\mathbf{1}_L * \psi(q))(x) = \operatorname{vol}(L) \mathbf{1}_{L^\#}(x) \psi(q(x)).$$

Consequently the oscillatory integral stabilizes on all sufficiently large lattices $N \supset L^\#$, and its phase is the Weil index. More precisely,

$$(1) \quad \Gamma_\psi(q) = |L^\#/L|^{-1/2} \sum_{x \in L^\#/L} \psi(q(x)).$$

Nondegeneracy of the polar character gives the sum absolute value $|L^\#/L|^{1/2}$. Formula (1) is independent of the sufficiently small lattice chosen. It is also a finite computation in residue characteristic two, where factors of 2 must be retained in B_q .

For an explicit odd-residue-characteristic formula, assume ψ is trivial on $\mathfrak{o}$ and nontrivial on $\varpi^{-1}\mathfrak{o}$. Put $Q = |\mathfrak{o}/\varpi\mathfrak{o}|$ and

$$\bar{\psi}(\bar{t}) = \psi(\varpi^{-1}t).$$

For $a = \varpi^v u$, with $u \in \mathfrak{o}^\times$, square scaling reduces the calculation to $v = 0$ or 1 , and gives

$$\Gamma_\psi(ax^2) = \begin{cases} 1, & v \text{ even,} \\ Q^{-1/2} \sum_{t \in \mathfrak{o}/\varpi\mathfrak{o}} \bar{\psi}(\bar{u}t^2), & v \text{ odd.} \end{cases}$$

Indeed the even case uses the lattice $\mathfrak{o}$, whereas for $a = \varpi u$ the stabilized integral over $\varpi^{-1}\mathfrak{o}$ is the displayed Gauss sum before normalization. General forms are computed by diagonalizing and multiplying these one-dimensional indices. For a character of a different conductor, write $\psi(t) = \psi_0(ct)$ with ψ_0 of conductor $\mathfrak{o}$, and apply the formula to cq .

Lemma 7.1 (Reduced norm). *Let A be a central simple algebra of degree r over k . There is a unique polynomial map $\text{Nrd}_{A/k} : A \rightarrow k$ whose scalar extension to a splitting field K is the determinant under $A \otimes_k K \simeq M_r(K)$. On $A^\times$ it is a homomorphism to $k^\times$.*

Proof. The determinant is unchanged by changing the splitting isomorphism, since automorphisms of the matrix algebra are inner. Its polynomial coefficients therefore descend to k . Multiplicativity follows from that of the determinant, and uniqueness follows after scalar extension. $\square$

Proposition 7.2 (Quaternion norm). *Let D be a quaternion algebra over k , with canonical involution $z \mapsto \bar{z}$ and reduced norm $n(z) = z\bar{z}$. Then*

$$\Gamma_\psi(n) = \begin{cases} -1, & D \text{ is a division algebra,} \\ 1, & D \simeq M_2(k). \end{cases}$$

Proof. Write $D = (a, b)_k$, with $i^2 = a$, $j^2 = b$, and $ij = -ji$. Its norm form is

$$n \simeq \langle 1, -a, -b, ab \rangle.$$

Put $g_\psi(c) = \Gamma_\psi(cx^2)$ and $\gamma_\psi(c) = g_\psi(c)/g_\psi(1)$. The one-dimensional Weil-index relation is

$$\gamma_\psi(a)\gamma_\psi(b) = (a, b)_k \gamma_\psi(ab),$$

where $(a, b)_k$ is the quadratic Hilbert symbol; see [Kud, Chapter I, Lemma 4.1]. Since $g_\psi(-c) = g_\psi(c)^{-1}$, multiplicativity yields

$$\Gamma_\psi(n) = \frac{g_\psi(1)g_\psi(ab)}{g_\psi(a)g_\psi(b)} = (a, b)_k.$$

Over a local field the Hilbert symbol is -1 precisely for the division quaternion algebra. In the split case the norm is the determinant on $M_2(k)$, an orthogonal sum of two hyperbolic planes, which also gives index 1. Over $\mathbb{R}$ the division norm is positive definite of dimension four, and the real formula independently gives -1 . $\square$

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