

HARTOG'S LEMMA IN ALGEBRAIC GEOMETRY

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1. INTRODUCTION

We introduce the algebraic Hartog's lemma. This was inspired by an exercise in Gathmann's notes and delivered in Professor Voronov's lecture and office hour discussions. We recall that in Gathmann's notes, the following statement holds.

Proposition 1.1. *Let X be an affine variety and let $A(X)$ be its coordinate ring. Assume that $A(X)$ is a unique factorization domain. If $Y \subsetneq X$ is a nonempty irreducible closed subvariety and $U = X \setminus Y$, then the restriction map*

$$A(X) \longrightarrow H^0(U, \mathcal{O}_U)$$

is an isomorphism if and only if $\text{codim}_X Y \geq 2$. [1, Exercise 3.12(a)]

Proof. Write $A = A(X)$ and $K = \text{Frac}(A)$. Since A is a domain and U is nonempty, a regular function on U determines an element of K , and the restriction map is injective.

Suppose that $\text{codim}_X Y \geq 2$, and let $f = a/b \in H^0(U, \mathcal{O}_U)$, with $a, b \in A$ relatively prime and $b \neq 0$. If b is not a unit, choose an irreducible factor p of b . In a UFD, (p) is a height-one prime. The corresponding point $\eta \in \text{Spec } A$ does not belong to Y , because Y has codimension at least two. Hence $f \in A_{(p)}$. We may therefore write $a/b = c/s$ with $s \notin (p)$. The equality $as = bc$ implies $p \mid as$, so $p \mid a$, contrary to the choice of a, b . Thus b is a unit and $f \in A$.

Conversely, if $\text{codim}_X Y = 1$, then the prime ideal defining Y is (p) for some irreducible $p \in A$. Thus $U = D(p)$, and $1/p$ is regular on U but does not belong to A . Since Y is nonempty and proper, these are all possible cases. $\square$

2. DEPTH AND LOCAL COHOMOLOGY

Lemma 2.1. *Let A be a Noetherian ring and let I be an ideal of A . M is a finitely generated A -module such that $IM \neq M$. The following numbers identifies:*

- (1) : $\text{depth}_I M = i$,
- (2) : i is the smallest integer such that $\text{Ext}_A^i(A/I, M) \neq 0$.
- (3) : i is the smallest integer such that $H_I^i(M) \neq 0$. [2, Tag 0AVY]

Proof. Recall that $\text{depth}_I M$ is the maximum length of an M -regular sequence contained in I . Such lengths are finite: choose $\mathfrak{q} \in V(I) \cap \text{Supp } M$, which exists because $IM \neq M$, and localize at $\mathfrak{q}$. Any such sequence remains regular after localization, with nonzero successive quotients by Nakayama's lemma, so its length is at most $\dim A_{\mathfrak{q}}$.

Choose a maximal M -regular sequence $x_1, \dots, x_r$ in I and put

$$N = M/(x_1, \dots, x_r)M.$$

Maximality implies that every element of I is a zero divisor on N . The zero divisors on a finite module form the union of its finitely many associated primes. Prime avoidance therefore gives $I \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in \text{Ass}_A N$. An element of N with annihilator $\mathfrak{p}$ is annihilated by I , so

$$\text{Hom}_A(A/I, N) \neq 0.$$

Let L be a finite module with $IL \neq L$, and let $x \in I$ be L -regular. Applying $\text{Ext}_A^\bullet(A/I, -)$ to

$$0 \longrightarrow L \xrightarrow{x} L \longrightarrow L/xL \longrightarrow 0$$

gives, for $j \geq 0$,

$$0 \longrightarrow \operatorname{Ext}_A^j(A/I, L) \longrightarrow \operatorname{Ext}_A^j(A/I, L/xL) \longrightarrow \operatorname{Ext}_A^{j+1}(A/I, L) \longrightarrow 0.$$

Indeed, multiplication by x on these Ext groups is zero because $x(A/I) = 0$. Also $\operatorname{Hom}_A(A/I, L) = 0$, since x is injective on L . It follows from these short exact sequences that the first nonzero Ext degree for L is one greater than that for L/xL . Starting with N and proceeding backwards through $x_r, \dots, x_1$, we obtain

$$\operatorname{Ext}_A^j(A/I, M) = 0 \quad (j < r), \quad \operatorname{Ext}_A^r(A/I, M) \neq 0.$$

In particular, every maximal regular sequence has the same length r , so $r = \operatorname{depth}_I M$. This proves (1) = (2). The same argument gives the regular-element formula

$$\operatorname{depth}_I(L/xL) = \operatorname{depth}_I L - 1$$

whenever $IL \neq L$ and $x \in I$ is L -regular.

For local cohomology, recall that

$$H_I^j(L) = \varinjlim_n \operatorname{Ext}_A^j(A/I^n, L).$$

This follows by applying an injective resolution to $\Gamma_I(L) = \varinjlim_n \operatorname{Hom}_A(A/I^n, L)$ and using exactness of filtered direct limits. In particular, every element of $H_I^j(L)$ is annihilated by a power of I .

We now induct on $r = \operatorname{depth}_I M$. If $r = 0$, the associated-prime argument above gives a nonzero element annihilated by I , so $H_I^0(M) \neq 0$. If $r > 0$, choose an M -regular $x \in I$ and set $\overline{M} = M/xM$. The regular-element formula and induction give

$$H_I^j(\overline{M}) = 0 \quad (j < r - 1), \quad H_I^{r-1}(\overline{M}) \neq 0.$$

The long exact sequence for local cohomology shows that multiplication by x is injective on $H_I^j(M)$ for every $j < r$; for $j = 0$ use the beginning of that sequence. Since every element of these modules is killed by a power of x , injectivity forces $H_I^j(M) = 0$ for $j < r$. The same sequence then gives an injection

$$0 \neq H_I^{r-1}(\overline{M}) \hookrightarrow H_I^r(M).$$

Thus $H_I^r(M) \neq 0$, proving (1) = (3). $\square$

We will also use the localization formula

$$(2.1) \quad \operatorname{depth}_I M = \min_{\mathfrak{p} \in V(I) \cap \operatorname{Supp} M} \operatorname{depth}_{A_{\mathfrak{p}}} M_{\mathfrak{p}}.$$

To see this, use $x_1, \dots, x_r$ and N from the preceding proof. At each prime in the indicated set the sequence remains regular, so the local depth is at least r . On the other hand, choose $\mathfrak{p} \in \operatorname{Ass}_A N$ containing I . Then $\operatorname{depth}_{A_{\mathfrak{p}}} N_{\mathfrak{p}} = 0$, and repeated application of the regular-element formula gives $\operatorname{depth}_{A_{\mathfrak{p}}} M_{\mathfrak{p}} = r$. This proves the formula. We use the convention $\operatorname{depth}(0) = +\infty$.

3. SERRE'S CRITERION FOR NORMALITY

Definition 3.1. Let R be a Noetherian ring and let $k \geq 0$ be an integer.

(R_k) : We say R has property (R_k) if for any prime ideal $\mathfrak{p}$ of height less or equal to k , $R_{\mathfrak{p}}$ is a regular ring.

(S_k) : We say R has property (S_k) if for every prime $\mathfrak{p}$, the local ring $R_{\mathfrak{p}}$ has depth at least $\min\{k, \dim(R_{\mathfrak{p}})\}$.

Theorem 3.2 (Serre's criterion for normality). *Suppose R is Noetherian, then R is normal if and only if R satisfies $(S_2) + (R_1)$.*

Proof. Suppose first that R is normal. We work locally with a Noetherian integrally closed local domain $(A, \mathfrak{m})$.

We first observe that if $t \in \operatorname{Frac}(A) \setminus A$ satisfies $t\mathfrak{m} \subseteq A$, then $\mathfrak{m}$ is principal. Indeed, if $t\mathfrak{m} \subseteq \mathfrak{m}$, choose generators $u_1, \dots, u_s$ of the nonzero ideal $\mathfrak{m}$ and write $tu_i = \sum_j c_{ij}u_j$ with $c_{ij} \in A$. Multiplication by the adjugate matrix shows that $\det(t\operatorname{Id} - (c_{ij}))$ annihilates every u_i . Since A is

a domain and $\mathfrak{m} \neq 0$, this determinant is zero. It is a monic equation for t over A , contradicting integral closedness. Thus there is $x \in \mathfrak{m}$ such that $tx \in A \setminus \mathfrak{m}$ is a unit. For every $y \in \mathfrak{m}$,

$$y = x \frac{ty}{tx} \in xA,$$

so $\mathfrak{m} = xA$.

If $\dim A \geq 2$, choose $0 \neq a \in \mathfrak{m}$. If $\mathfrak{m}$ were associated to A/aA , there would be $b \notin aA$ with $\mathfrak{m}b \subseteq aA$. Then $t = b/a \notin A$ would satisfy $t\mathfrak{m} \subseteq A$. The observation would make $\mathfrak{m}$ principal, contradicting $\dim A \geq 2$ by the principal ideal theorem. Hence $\mathfrak{m} \notin \text{Ass}_A(A/aA)$, and prime avoidance supplies an element of $\mathfrak{m}$ regular on A/aA . Together with a this is an A -regular sequence of length two. In dimension one, any nonzero element of $\mathfrak{m}$ is regular. This proves (S_2) .

If $\dim A = 1$, choose $0 \neq a \in \mathfrak{m}$. Since $\sqrt{aA} = \mathfrak{m}$, there is a least $n \geq 1$ such that $\mathfrak{m}^n \subseteq aA$. Choose $b \in \mathfrak{m}^{n-1} \setminus aA$. Again $t = b/a \notin A$ satisfies $t\mathfrak{m} \subseteq A$, so the observation shows that $\mathfrak{m}$ is principal. Thus A is regular. In dimension zero, A is a field. This proves (R_1) .

Conversely, suppose that R satisfies (R_1) and (S_2) . The (S_1) condition implies that every associated prime of R is minimal: at an associated prime $\mathfrak{p}$ the local depth is zero, so (S_1) forces $\dim R_{\mathfrak{p}} = 0$. Moreover, (R_0) makes $R_{\mathfrak{p}}$ a field at every minimal prime. Consequently R is reduced. Indeed, if the nilradical were a nonzero ideal, one of its associated primes would be an associated prime of R , hence a minimal prime at which that ideal remained nonzero. This contradicts the fact that the corresponding localization is a field.

For a reduced Noetherian ring its total ring of fractions is

$$Q(R) = \prod_{\mathfrak{q} \in \text{Min}(R)} \text{Frac}(R/\mathfrak{q}).$$

We show that R is integrally closed in $Q(R)$. Let $t \in Q(R)$ be integral over R and put $B = R[t]$, a finite R -algebra. If $B/R \neq 0$, choose $\mathfrak{p} \in \text{Ass}_R(B/R)$. When $\text{ht } \mathfrak{p} \leq 1$, the ring $R_{\mathfrak{p}}$ is either a field or a DVR, so it is integrally closed. Thus $B_{\mathfrak{p}} = R_{\mathfrak{p}}$, a contradiction. Here we use that localization of $Q(R)$ at $\mathfrak{p}$ is $Q(R_{\mathfrak{p}})$: in the displayed product, localization kills precisely the factors indexed by $\mathfrak{q} \not\subseteq \mathfrak{p}$, leaving the fraction fields corresponding to the minimal primes of $R_{\mathfrak{p}}$.

We must therefore have $\text{ht } \mathfrak{p} \geq 2$. Set $A = R_{\mathfrak{p}}$ and $D = B_{\mathfrak{p}} \subseteq Q(A)$. By (S_2) , choose an A -regular sequence x, y in its maximal ideal. Since $\mathfrak{p}$ is associated to B/R , there is $d \in D \setminus A$ whose class in D/A is annihilated by that maximal ideal. Thus $xd = a$ and $yd = b$ for some $a, b \in A$, and

$$ya = xb.$$

As y is regular modulo xA , we have $a = xc$ for some $c \in A$. Hence $x(d - c) = 0$. But x is a nonzerodivisor on A , so it is a unit in $Q(A)$ and is injective on D . Therefore $d = c \in A$, a contradiction. We conclude that $B = R$, proving integral closedness in $Q(R)$.

Finally, the coordinate idempotents of the finite product $Q(R)$ satisfy $e^2 - e = 0$, so they belong to R . They decompose R as a finite product of domains. Each factor is integrally closed in its fraction field, since R is integrally closed in $Q(R)$. Every local ring of R is therefore an integrally closed domain, and R is normal. $\square$

4. ALGEBRAIC HARTOG'S LEMMA-PRIMARY

Lemma 4.1. *Let R be a Noetherian domain satisfying (S_2) property, then we have the following equality inside $K = \text{Frac}(R)$:*

$$R = \bigcap_{\text{ht}(\mathfrak{p})=1} R_{\mathfrak{p}}.$$

Proof. The inclusion from left to right is clear. Conversely, suppose that $f = a/b \in K \setminus R$, with $a, b \in R$ and $b \neq 0$. Then the class $\bar{a} \in R/bR$ is nonzero. Choose

$$\mathfrak{p} \in \text{Ass}_R(R\bar{a}) \subseteq \text{Ass}_R(R/bR).$$

Since $(R\bar{a})_{\mathfrak{p}} \neq 0$, the element $\bar{a}$ remains nonzero after localization, so $a \notin bR_{\mathfrak{p}}$ and $f \notin R_{\mathfrak{p}}$. On the other hand,

$$\text{depth}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/bR_{\mathfrak{p}}) = 0.$$

The element b is a nonzerodivisor, so the regular-element formula gives $\text{depth } R_{\mathfrak{p}} = 1$. The (S_2) condition implies $\text{ht } \mathfrak{p} \leq 1$. Since $0 \neq b \in \mathfrak{p}$ and R is a domain, $\text{ht } \mathfrak{p} = 1$. Thus every $f \notin R$ fails to belong to some height-one localization, proving the reverse inclusion. If R is a field, the empty intersection is taken inside $K = R$. $\square$

Let $X = \text{Spec } A$ and suppose A is a Noetherian domain with (S_2) property, then for every closed subset $Z \subseteq X$ of codimension at least two, restriction induces an isomorphism

$$A \xrightarrow{\sim} H^0(X \setminus Z, \mathcal{O}_X).$$

Indeed, $U = X \setminus Z$ contains the generic point of X , so a section on U determines a rational function $f \in \text{Frac}(A)$. Every height-one point belongs to U , and hence $f \in A_{\mathfrak{p}}$ for every height-one prime $\mathfrak{p}$. The preceding lemma gives $f \in A$. Its associated global section agrees with the original section on U , since both have the same image in the function field. Uniqueness follows from the injection $A \hookrightarrow \text{Frac}(A)$.

5. ALGEBRAIC HARTOG'S LEMMA-GENERAL

We prove the generalized version of Hartog's lemma:

Theorem 5.1. *Let X be a Noetherian scheme, and let Z be a closed subscheme of X such that $\text{codim}_X Z \geq 2$, and for every $z \in Z$, the ring $\mathcal{O}_{X,z}$ satisfies (S_2) condition. Let $U = X \setminus Z$, then for any coherent sheaf $\mathcal{F}$ on X such that*

$$\text{depth}_{\mathcal{O}_{X,z}} \mathcal{F}_z \geq 2 \quad (z \in Z \cap \text{Supp } \mathcal{F}),$$

we have $H^0(U, \mathcal{F}|_U) \simeq H^0(X, \mathcal{F})$. In particular, any regular function defined on U extends uniquely to a regular function defined on the entire space X .

Proof. Let $V = \text{Spec } A \subset X$ be an affine open subscheme of X . Suppose $\mathcal{F}|_V = \widetilde{M}$, where M is a finitely generated A -module. Let $V(I) = V \cap Z$, where I is an ideal of A .

Consider the cohomology of support on $Z \cap V$, write $U \cap V = U'$ we have the long exact sequence

$$0 \rightarrow H_{Z \cap V}^0(V, \mathcal{F}) \rightarrow H^0(V, \mathcal{F}) \rightarrow H^0(U', \mathcal{F}|_{U'}) \rightarrow H_{Z \cap V}^1(V, \mathcal{F}) \rightarrow \cdots.$$

In this affine case this is exactly the cohomology defined on modules, i.e., $H_{Z \cap V}^i(V, \mathcal{F}) \simeq H_I^i(M)$.

If $IM \neq M$, the stalk condition and (2.1) give

$$\text{depth}_I M = \min_{\mathfrak{p} \in V(I) \cap \text{Supp } M} \text{depth}_{A_{\mathfrak{p}}} M_{\mathfrak{p}} \geq 2.$$

By Lemma 2.1, $H_I^i(M) = 0$ for $i = 0, 1$. If $IM = M$, then $V(I) \cap \text{Supp } M = \emptyset$ by Nakayama's lemma. Thus $I + \text{Ann}_A M = A$, and there is $u \in I$ acting as the identity on M . It acts as the identity on every $H_I^i(M)$ as well, but every element of those groups is killed by a power of u . Hence all these groups vanish in this case too. Consequently the restriction map is an isomorphism

$$H^0(V, \mathcal{F}) \xrightarrow{\sim} H^0(U', \mathcal{F}|_{U'}).$$

Now consider suppose $\{V_i = \text{Spec } A_i\}_{i \in I}$ is an affine cover of X , since X is Noetherian, X is quasi-compact, hence we could assume I is a finite index set. Given $s \in H^0(U, \mathcal{F}|_U)$, the affine isomorphisms give unique sections $s_i \in H^0(V_i, \mathcal{F})$ extending $s|_{U \cap V_i}$.

To check compatibility, cover each $V_i \cap V_j$ by affine open subsets W . On $W \cap U$, the two sections $s_i|_W$ and $s_j|_W$ both restrict to $s|_{W \cap U}$. Applying the affine injectivity already proved to W gives $s_i|_W = s_j|_W$. Thus s_i and s_j agree on all of $V_i \cap V_j$. By the sheaf axiom, the s_i glue to a section $\tilde{s} \in H^0(X, \mathcal{F})$ extending s . If two global sections extend s , affine injectivity on each V_i makes them equal. This proves the asserted isomorphism.

Finally, for $\mathcal{F} = \mathcal{O}_X$, the codimension assumption gives $\dim \mathcal{O}_{X,z} \geq 2$ for every $z \in Z$. The (S_2) assumption therefore yields

$$\text{depth } \mathcal{O}_{X,z} \geq \min\{2, \dim \mathcal{O}_{X,z}\} = 2.$$

Thus the required condition on $\mathcal{F}$ holds, proving the assertion about regular functions. The same reasoning applies to any locally free coherent sheaf, whose nonzero stalks are finite free modules of the same depth as the local ring. $\square$

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