MATH 8270: Abelian Varieties and their Moduli

1. Motivation

1.1. **Arithmetic of Modular Forms.** Let $\mathcal{H} := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ be the upper half-plane. The group

$$SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z},\ ad - bc = 1 \right\}$$

acts on $\mathcal{H}$ by Möbius transformations $\gamma z = (az + b)/(cz + d)$. Let Γ be a *congruence subgroup* of $SL_2(\mathbb{Z})$, meaning that Γ contains

$$\Gamma(N) := \ker\big(SL_2(\mathbb{Z}) \longrightarrow SL_2(\mathbb{Z}/N\mathbb{Z})\big)$$

for some integer $N \geq 1$.

Definition 1.1. A **holomorphic modular form** of weight $k \in \mathbb{Z}$ and level Γ is a holomorphic function $f : \mathcal{H} \to \mathbb{C}$ satisfying:

(1) : (automorphy condition) For every $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$,

$$f(\gamma z) = (cz + d)^k f(z).$$

(2) : (growth condition) For every $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, the function

$$(f|_k\gamma)(z) := (cz + d)^{-k} f(\gamma z)$$

is bounded as $\operatorname{Im} z \to \infty$, with $\operatorname{Re} z$ in a bounded interval. Equivalently, f is holomorphic at every cusp.

Remark 1.2. The form f is called a **cusp form** if condition (2) is replaced by

$(2)'$: (cuspidal condition) For every $\gamma \in SL_2(\mathbb{Z})$,

$$(f|_k\gamma)(z) \longrightarrow 0 \qquad \text{as } \operatorname{Im} z \longrightarrow \infty.$$

Write $M_k(\Gamma, \mathbb{C})$ for the space of modular forms of weight k and level Γ, and $S_k(\Gamma, \mathbb{C})$ for its subspace of cusp forms. For $q = e^{2\pi i z}$, a fundamental example is

$$\Delta(z) = \sum_{n>0} \tau(n) q^n = q \prod_{n>0} (1 - q^n)^{24} = q - 24q^2 + 252q^3 + \cdots \in S_{12}(SL_2(\mathbb{Z}), \mathbb{C}).$$

Here $\tau(n)$ is Ramanujan's tau function. Its coefficients satisfy

$$\tau(mn) = \tau(m)\tau(n) \qquad \text{if } \gcd(m, n) = 1, \tag{1}$$

$$\tau(p^{r+1}) = \tau(p)\tau(p^r) - p^{11}\tau(p^{r-1}) \qquad \text{if } p \text{ is prime and } r \geq 1. \tag{2}$$

Conjecture 1.3 (Ramanujan's conjecture)**.** *For every prime p,*

$$|\tau(p)| \leq 2p^{11/2}.$$

Deligne proved this conjecture using the Weil conjectures and étale cohomology over finite fields, building on earlier work of Kuga, Sato, Shimura, and Ihara.

The analytic definition of modular forms has an algebro-geometric interpretation. For simplicity in this discussion, take $\Gamma = \Gamma(N)$ with $N \geq 3$. It acts freely on $\mathcal{H}$, and

$$Y_{\Gamma,\mathbb{C}} := \Gamma\backslash\mathcal{H} \hookrightarrow X_{\Gamma,\mathbb{C}} := \Gamma\backslash\mathcal{H}^*, \qquad \mathcal{H}^* := \mathcal{H} \cup \mathbb{P}^1(\mathbb{Q}),$$

are a modular curve and its compactification. Let D be the reduced divisor of cusps. The Hodge line bundle ω on $Y_{\Gamma,\mathbb{C}}$ has fiber $H^0(E, \Omega^1_E)$ at the point represented by an elliptic curve E with level structure. Its canonical extension to $X_{\Gamma,\mathbb{C}}$ satisfies

$$M_k(\Gamma, \mathbb{C}) \simeq H^0(X_{\Gamma,\mathbb{C}}, \omega^{\otimes k}),$$

and

$$S_k(\Gamma, \mathbb{C}) \simeq H^0(X_{\Gamma,\mathbb{C}}, \omega^{\otimes k}(-D)).$$

These identifications come from the transformation law and the conditions at the cusps. The compact Riemann surface $X_{\Gamma,\mathbb{C}}$ is algebraic, and GAGA identifies its analytic line bundles and their cohomology with their algebraic counterparts.

Theorem 1.4 (Serre – GAGA)**.** *Let X be a proper scheme over $\mathbb{C}$, and let X^{an} denote its associated complex analytic space. The analytification functor*

$$(\cdot)^{\mathrm{an}} : \mathrm{Coh}(X) \longrightarrow \mathrm{Coh}(X^{\mathrm{an}})$$

has the following properties:

(1) *(**Equivalence of categories**) It induces an equivalence*

$$\mathrm{Coh}(X) \xrightarrow{\sim} \mathrm{Coh}(X^{\mathrm{an}}).$$

(2) *(**Cohomology comparison**) For every coherent sheaf $\mathcal{F}$ and every $i \geq 0$, the natural map*

$$H^i(X, \mathcal{F}) \longrightarrow H^i(X^{\mathrm{an}}, \mathcal{F}^{\mathrm{an}})$$

is an isomorphism.

(3) *(**Algebraicity of morphisms**) If Y is another proper scheme over $\mathbb{C}$, then*

$$\mathrm{Hom}_{\mathrm{Sch}/\mathbb{C}}(X, Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{An}}(X^{\mathrm{an}}, Y^{\mathrm{an}}).$$

The spaces of modular forms carry Hecke operators T_p for primes p not dividing the level. The good-prime Hecke operators commute; on cusp forms they admit simultaneous eigenforms. Their eigenvalues encode arithmetic information, as the coefficients $\tau(p)$ illustrate.

On the other hand, for $k \geq 2$ the Eichler–Shimura isomorphism relates modular forms to cohomology:

$$M_k(\Gamma, \mathbb{C}) \oplus \overline{S_k(\Gamma, \mathbb{C})} \simeq H^1(Y_{\Gamma,\mathbb{C}}, \mathbb{V}_{k-2,\mathbb{C}}).$$

Here $\mathbb{V}_{k-2,\mathbb{C}}$ is the local system associated with the Γ-representation $\mathrm{Sym}^{k-2}(\mathbb{C}^2)$. Equivalently, if $\pi : \mathcal{E} \to Y_{\Gamma,\mathbb{C}}$ is the universal elliptic curve, it can be identified with $\mathrm{Sym}^{k-2}(R^1\pi_*\mathbb{C})$. Its fibers are isomorphic to $\mathrm{Sym}^{k-2}(\mathbb{C}^2)$; the local system need not be constant. The cuspidal part is

$$S_k(\Gamma, \mathbb{C}) \oplus \overline{S_k(\Gamma, \mathbb{C})} \simeq H^1_{\mathrm{par}}(Y_{\Gamma,\mathbb{C}}, \mathbb{V}_{k-2,\mathbb{C}}),$$

where H^1_{par} is the image of compactly supported cohomology in ordinary cohomology.

The curves and their moduli interpretation have integral models over $\mathbb{Z}[\zeta_N, 1/N]$. Fix an embedding of their field of definition into $\mathbb{C}$. For every prime ℓ, comparison with étale cohomology gives

$$H^1(Y^{\mathrm{an}}_{\Gamma,\mathbb{C}}, \mathbb{V}_{k-2,\mathbb{Q}_\ell}) \simeq H^1_{\text{ét}}(Y_{\Gamma,\overline{\mathbb{Q}}}, \mathbb{V}_{k-2,\ell}),$$

where $\mathbb{V}_{k-2,\ell} := \mathrm{Sym}^{k-2}(R^1\pi_*\mathbb{Q}_\ell)$ on the algebraic modular curve. Thus the right-hand side carries a Galois action compatible with Hecke operators. At a prime of good reduction of residue characteristic $p \neq \ell$, proper smooth base change gives, for example,

$$H^1_{\text{ét}}(X_{\Gamma,\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) \simeq H^1_{\text{ét}}(X_{\Gamma,\overline{\mathbb{F}}_p}, \mathbb{Q}_\ell),$$

after choosing a specialization. For higher weights, the relevant coefficient systems can be realized in the cohomology of powers of the universal elliptic curve. One must account for the compactification and its boundary: proper smooth base change cannot be applied directly to the nonproper curve Y_Γ.

2. Abelian Varieties: General theory

In this section, we develop the basic geometry of abelian schemes, beginning with their functorial description and rigidity. Our main reference is [1, Sec. 1.3].

2.1. **Category Theory.** Throughout these notes, $\mathcal{C}$ denotes a category. We write Set for the category of sets, Grp for the category of groups, Sch for the category of schemes, and Sch_S for the category of schemes over a fixed scheme S. The opposite category $\mathcal{C}^{\mathrm{op}}$ has the same objects as $\mathcal{C}$ and all arrows reversed. For $X \in \mathcal{C}$, define

$$h_X := \mathrm{Hom}_{\mathcal{C}}(-, X) : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{Set}.$$

Definition 2.1. A functor $F : \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$ is *representable* if $F \simeq h_X$ for some $X \in \mathcal{C}$.

Theorem 2.2 (Yoneda lemma)**.** *For every functor $F : \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$, evaluation at Id_X gives a bijection*

$$\mathrm{Nat}(h_X, F) \xrightarrow{\sim} F(X).$$

In particular,

$$\mathrm{Nat}(h_X, h_Y) \simeq \mathrm{Hom}_{\mathcal{C}}(X, Y).$$

2.2. **Group Schemes.**

Definition 2.3. A *group functor* over S is a functor $G : \mathsf{Sch}_S^{\mathrm{op}} \to \mathsf{Grp}$. It is commutative if $G(T)$ is abelian for every S-scheme T.

Remark 2.4. A morphism $\phi : G \to H$ of group functors is a natural transformation whose components $\phi(T) : G(T) \to H(T)$ are group homomorphisms. In particular, these maps must commute with pullback along every morphism of S-schemes.

Definition 2.5. A *group scheme* over S is a group functor whose underlying set-valued functor

$$\mathsf{Sch}_S^{\mathrm{op}} \longrightarrow \mathsf{Grp} \longrightarrow \mathsf{Set}$$

is represented by an S-scheme, also denoted G.

Remark 2.6. The functor $\mathsf{Grp} \to \mathsf{Set}$ above is the forgetful functor.

By the Yoneda lemma, a homomorphism of group schemes is equivalently an S-morphism of their representing schemes whose maps on all functorial points are group homomorphisms. We now describe the identity, inverse, and multiplication morphisms explicitly.

Remark 2.7. (1) : The identity $e_S \in G(S)$ defines a section $e : S \to G$ of the structure morphism $\pi : G \to S$. For any $f : T \to S$, the identity of $G(T)$ is represented by $e \circ f$.

(2) : The inverse of $\mathrm{Id}_G \in G(G)$ defines an S-morphism $[-1] : G \to G$. Naturality implies that the inverse of $x : T \to G$ is $[-1] \circ x$.

(3) : The product of the two projections in $G(G \times_S G)$ defines a morphism $m : G \times_S G \to G$. For $x, y \in G(T)$, their product is the composite

$$T \xrightarrow{(x,y)} G \times_S G \xrightarrow{m} G.$$

The group axioms translate into the following identities of morphisms:

(1) : (associativity) The diagram

$$\begin{array}{ccc} G \times_S G \times_S G & \xrightarrow{\mathrm{Id}_G \times m} & G \times_S G \\ {\scriptstyle m \times \mathrm{Id}_G}\downarrow & & \downarrow{\scriptstyle m} \\ G \times_S G & \xrightarrow{\quad m \quad} & G \end{array}$$

commutes.

(2) : (identity) We have

$$m \circ (\mathrm{Id}_G, e \circ \pi) = \mathrm{Id}_G = m \circ (e \circ \pi, \mathrm{Id}_G).$$

(3) : (inverse) We have

$$m \circ (\mathrm{Id}_G, [-1]) = e \circ \pi = m \circ ([-1], \mathrm{Id}_G).$$

(4) : A homomorphism $\phi : G \to H$ respects e, $[-1]$, and m; in particular,

$$\phi \circ m_G = m_H \circ (\phi \times \phi).$$

(5) : The group scheme is commutative exactly when $m = m \circ s$, where $s = (\mathrm{pr}_2, \mathrm{pr}_1)$ swaps the factors of $G \times_S G$.

Conversely, an S-scheme G equipped with morphisms

$$\begin{aligned} m &: G \times_S G \to G, \\ [-1] &: G \to G, \\ e &: S \to G \end{aligned}$$

satisfying the group axioms above defines a group functor $T \mapsto \mathrm{Hom}_S(T, G)$. Thus the functorial and geometric definitions agree.

If $S = \operatorname{Spec} k$ for a field k and G is a variety over k, we call G a *group variety*.

2.3. **Elliptic curves.**

Definition 2.8. An *elliptic curve* over a field k is a smooth, proper, geometrically connected curve E of genus 1, together with a point $P_0 \in E(k)$.

Remark 2.9. For a smooth proper curve, connectedness together with a k-rational point implies geometric connectedness. Indeed, the finite étale k-algebra $H^0(E, \mathcal{O}_E)$ is a field when E is connected, and evaluation at the rational point embeds this field into k, so it equals k.

Lemma 2.10. *Let D be an effective divisor on E, and write $l(D) = \dim_k H^0(E, \mathcal{O}_E(D))$.*

(a) : The line bundle $\mathcal{O}_E(D)$ is globally generated if and only if

$$l(D_{\overline{k}} - x) < l(D_{\overline{k}})$$

for every $x \in \mathrm{Supp}(D_{\overline{k}})$. Outside this support, the section 1 already has nonzero value.

(b) : The line bundle $\mathcal{O}_E(D)$ is very ample if and only if, for all $x, y \in E(\overline{k})$, including $x = y$,

$$l(D_{\overline{k}} - x - y) < l(D_{\overline{k}} - x) < l(D_{\overline{k}}).$$

Corollary 2.11. *The line bundle $\mathcal{O}_E(nP_0)$ is very ample for $n \geq 3$. In particular, a basis of $H^0(E, \mathcal{O}_E(3P_0))$ defines a closed immersion $E \hookrightarrow \mathbb{P}^2_k$.*

Proof. By Riemann–Roch and the triviality of the canonical bundle of a genus-one curve, $l(D) = \deg D$ for every divisor of positive degree. For $D = nP_0$ with $n \geq 3$, the three dimensions in part (b) are $n-2$, $n-1$, and n. □

2.4. **Abelian Schemes and Abelian Varieties.**

Definition 2.12. An *abelian scheme* is a proper smooth group scheme $\pi : A \to S$ with connected fibers.

Remark 2.13. The fibers are automatically geometrically connected: each is a smooth proper connected group scheme over a field and has a rational identity point, so its ring of global functions is that field.

When $S = \operatorname{Spec} k$, we call A an **abelian variety** over k.

Remark 2.14. The morphism π is of finite presentation: it is locally of finite presentation by smoothness, and quasi-compact and quasi-separated by properness. Finite-presentation data descend along suitable inverse limits. Consequently, arguments compatible with this descent may be reduced, locally on S, to a Noetherian base, and often to a base of finite type over $\mathbb{Z}$; see [1, Thm. 1.3.1.3].

Let $e : S \to A$ be the identity section. Define

$$\underline{\mathrm{Lie}}^{\vee}_{A/S} := e^*\Omega^1_{A/S}, \qquad \underline{\mathrm{Lie}}_{A/S} := \mathrm{Hom}_{\mathcal{O}_S}(e^*\Omega^1_{A/S}, \mathcal{O}_S).$$

We will prove the following two results.

Proposition 2.15. *Evaluation at the identity induces an isomorphism*

$$e^* : \pi_*\Omega^i_{A/S} \xrightarrow{\sim} \bigwedge^i_{\mathcal{O}_S} \underline{\mathrm{Lie}}^{\vee}_{A/S}.$$

Proposition 2.16. *Every S-morphism $A \to A$ preserving the identity section is a group homomorphism. In particular, applying this to the inverse morphism shows that A is commutative.*

Remark 2.17. Commutativity was not imposed in our definition of an abelian scheme; it follows from Proposition 2.16.

If E/k is an abelian variety of dimension 1, Serre duality and Proposition 2.15 give

$$\dim_k H^1(E, \mathcal{O}_E) = \dim_k H^0(E, \Omega^1_E) = \dim_k e^*\Omega^1_E = 1.$$

Thus its underlying pointed curve is elliptic. Conversely, an elliptic curve (E, P_0) has a group law characterized by the Abel–Jacobi isomorphism

$$E \xrightarrow{\sim} \mathrm{Pic}^0_{E/k}, \qquad P \longmapsto \mathcal{O}_E(P - P_0).$$

Over an algebraic closure, Riemann–Roch shows that every degree-zero divisor class has a unique representative of this form; the resulting morphism and its inverse descend to k. The group law agrees with any other group scheme law having identity P_0, by Proposition 2.16. Hence there is a bijection

$$\{\text{elliptic curves over } k\}/\simeq \longleftrightarrow \{\text{abelian varieties of dimension 1 over } k\}/\simeq .$$

2.5. **Invariant differentials.** We first prove the statement about differentials underlying Proposition 2.15. Let $\pi : G \to S$ be a smooth group scheme. For an S-scheme S' and $g \in G(S')$, left translation on $G_{S'}$ is the composite

$$\begin{array}{ccc} G_{S'} & \xrightarrow{(g\circ\pi_{S'},\mathrm{Id})} & G_{S'} \times_{S'} G_{S'} \\ & {\scriptstyle T_g}\searrow \quad \swarrow{\scriptstyle m_{S'}} & \\ & G_{S'} & \end{array}$$

It is an S'-automorphism with inverse $T_{g^{-1}}$ and therefore acts on relative differential forms.

Definition 2.18. A section ω of $\pi_*\Omega^i_{G/S}$ is *left invariant* if, after every base change $S' \to S$, its pullback satisfies $T_g^*\omega_{S'} = \omega_{S'}$ for all $g \in G(S')$.

If two left invariant forms have the same value at e, they have the same value along every section g: use $T_g \circ e_{S'} = g$. Taking $S' = G$ and the universal section $g = \mathrm{Id}_G$ proves uniqueness.

Proposition 2.19. *For every section ω_0 of $e^*\Omega^i_{G/S}$, there is a unique left invariant section ω of $\pi_*\Omega^i_{G/S}$ satisfying $e^*\omega = \omega_0$.*

Proof. For $i = 1$, multiplication and the decomposition

$$\Omega^1_{G\times_S G/S} \simeq \mathrm{pr}_1^*\,\Omega^1_{G/S} \oplus \mathrm{pr}_2^*\,\Omega^1_{G/S}$$

give a canonical map

$$m^*\Omega^1_{G/S} \longrightarrow \mathrm{pr}_2^*\,\Omega^1_{G/S}.$$

Pull it back along $a = (\mathrm{Id}_G, e \circ \pi) : G \to G \times_S G$. Since $m \circ a = \mathrm{Id}_G$, this yields

$$\theta : \Omega^1_{G/S} \longrightarrow \pi^* e^* \Omega^1_{G/S}.$$

Along a section $g \in G(S')$, the map $g^*\theta$ is pullback by T_g evaluated at the identity. It is an isomorphism, since T_g is an automorphism. In particular, the universal section shows that θ is an isomorphism on G. Define $\omega = \theta^{-1}(\pi^*\omega_0)$. At a functorial point g, its value is transported from ω_0 by $T_{g^{-1}}$. The identity $T_g \circ T_h = T_{gh}$ shows that this construction is left invariant, and its value at e is ω_0. Taking exterior powers proves the assertion for every i. Uniqueness was established above. □

Corollary 2.20. *The canonical map defined by left translation is an isomorphism*

$$\pi^* e^* \Omega^i_{G/S} \xrightarrow{\sim} \Omega^i_{G/S}.$$

Proof. This is the inverse of $\bigwedge^i \theta$ constructed in the proof of Proposition 2.19. □

Corollary 2.21. *If $G \to S$ is smooth of relative dimension d, then, locally on S, the sheaf $\Omega^i_{G/S}$ is freely generated by $\binom{d}{i}$ left invariant forms.*

Corollary 2.22. *If $\mathcal{O}_S \xrightarrow{\sim} \pi_*\mathcal{O}_G$, then evaluation at the identity gives*

$$\pi_*\Omega^i_{G/S} \xrightarrow{\sim} e^*\Omega^i_{G/S}.$$

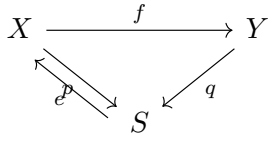
Proof. The sheaf $e^*\Omega^i_{G/S}$ is locally free. By Corollary 2.20 and the projection formula,

$$\pi_*\Omega^i_{G/S} \simeq \pi_*\pi^*(e^*\Omega^i_{G/S}) \simeq \pi_*\mathcal{O}_G \otimes_{\mathcal{O}_S} e^*\Omega^i_{G/S} \simeq e^*\Omega^i_{G/S}.$$

For an abelian scheme, properness, smoothness, and geometric connectedness of the fibers imply $\pi_*\mathcal{O}_A \simeq \mathcal{O}_S$, compatibly with base change. This proves Proposition 2.15. □

2.6. **Rigidity Lemma.**

Proposition 2.23. *Suppose we have a commutative diagram of locally Noetherian schemes*



satisfying:

(1) : *S is connected;*
(2) : *$p : X \to S$ is open and closed and has a section e;*
(3) : *$q : Y \to S$ is separated;*
(4) : *the canonical map $\mathcal{O}_S \to p_*\mathcal{O}_X$ is an isomorphism;*
(5) : *for some $s \in S$, the image $f(X_s)$ is set-theoretically a single point.*

Then $f = \eta \circ p$, where $\eta := f \circ e : S \to Y$.

Proof. Let $Z \hookrightarrow X$ be the equalizer of f and $\eta \circ p$; it is closed because q is separated. We first show that if $f(X_t)$ is set-theoretically a single point for some $t \in S$, then Z contains $p^{-1}(U)$ for an open neighborhood U of t.

Choose an affine open neighborhood $V \subset Y$ of that point. The closed subset $C := X \setminus f^{-1}(V)$ does not meet X_t. Since p is closed, $S \setminus p(C)$ is an open neighborhood of t. Choose an affine neighborhood U of t inside it. Then $f(X_U) \subset V$, and assumption (4) gives

$$\Gamma(X_U, \mathcal{O}_{X_U}) \simeq \Gamma(U, \mathcal{O}_U).$$

The morphism $X_U \to V$ is therefore induced by a ring map

$$\Gamma(V, \mathcal{O}_V) \longrightarrow \Gamma(U, \mathcal{O}_U),$$

so it factors through $p : X_U \to U$. Restriction along e identifies the factor with $\eta|_U$. This proves the claim.

Let U_0 be the union of the open subsets $U \subset S$ on which $f|_{X_U} = \eta \circ p$. The claim shows that U_0 is nonempty. If $t \notin p(X \setminus Z)$, then X_t lies set-theoretically in Z, so $f(X_t) = \{\eta(t)\}$ and the claim implies $t \in U_0$. Conversely, no point of U_0 belongs to $p(X \setminus Z)$. Thus

$$S \setminus U_0 = p(X \setminus Z),$$

which is open because p is open. Since S is connected, $U_0 = S$, proving the result. This is the rigidity lemma of [1, Prop. 1.3.1.4]. □

Remark 2.24. A flat morphism locally of finite presentation is open. In particular, smooth morphisms are open, while proper morphisms are universally closed. Thus the structure morphism of an abelian scheme is both open and closed. Flatness by itself does not imply local finite presentation or openness.

Corollary 2.25. *Let $p : A \to S$ be an abelian scheme over a connected scheme S, and let $q : G \to S$ be a separated group scheme of finite presentation. Suppose $f, g : A \to G$ are S-morphisms such that $f_s = g_s$ for some $s \in S$. Then there is a section $\eta : S \to G$ such that*

$$f = m_G \circ (\eta \circ p, g).$$

Proof. By finite-presentation descent we may reduce to a locally Noetherian base. Apply Proposition 2.23 to

$$x \longmapsto f(x)g(x)^{-1}.$$

Its restriction to A_s is the identity point. Thus it is $\eta \circ p$, which gives the required identity. The section is $\eta = m_G \circ (f \circ e_A, [-1]_G \circ g \circ e_A)$. □

Corollary 2.26. *Let $A \to S$ be an abelian scheme and $G \to S$ a separated group scheme of finite presentation. If an S-morphism $h : A \to G$ satisfies $h \circ e_A = e_G$, then h is a group homomorphism.*

Proof. We work locally on S, where the finite-presentation data descend to a Noetherian base. Define morphisms over A

$$f, g : A \times_S A \longrightarrow A \times_S G$$

by

$$f(x_1, x_2) = (x_1, h(x_1 x_2)), \qquad g(x_1, x_2) = (x_1, h(x_2)).$$

Use the first projections as structure morphisms:

$$\begin{array}{ccc} A \times_S A & \overset{f}{\underset{g}{\rightrightarrows}} & A \times_S G \\ & {\scriptstyle p} \searrow \quad \swarrow {\scriptstyle q} & \\ & A & \end{array}$$

The maps agree on the fiber over $e_A(s)$ for each $s \in S$. On every connected component of the Noetherian scheme A, the preceding corollary therefore gives a morphism $\eta : A \to G$ such that

$$h(x_1 x_2) = \eta(x_1) h(x_2).$$

Every such component meets the identity section: the fibers of $A \to S$ are connected, and its open and closed structure morphism identifies its connected components with those of S. Putting $x_2 = e_A$ yields $\eta(x_1) = h(x_1)$, and hence

$$h(x_1 x_2) = h(x_1) h(x_2).$$

By Yoneda, this is the required equality of morphisms. □

Remark 2.27. Taking $G = A$ proves Proposition 2.16.

Corollary 2.28. *Every abelian scheme $A \to S$ is commutative.*

Proof. By Corollary 2.26, the inverse morphism is a homomorphism. Thus, on functorial points,

$$(xy)^{-1} = x^{-1} y^{-1}.$$

The group axioms also give $(xy)^{-1} = y^{-1} x^{-1}$, so $xy = yx$. □

Corollary 2.29. *An abelian scheme has a unique group scheme structure with a prescribed identity section.*

Proof. Apply Corollary 2.26 to the identity map between any two such group scheme structures. It respects multiplication, so the two laws coincide. □

2.7. **Isogenies.**

Definition 2.30. An *isogeny* $f : G \to G'$ between smooth group schemes over S is a surjective group scheme homomorphism with quasi-finite kernel.

Definition 2.31. An isogeny $f : A \to A'$ of abelian schemes is an isogeny of smooth group schemes in the preceding sense.

Lemma 2.32. *An isogeny between smooth group schemes is flat. Consequently, an isogeny between proper smooth group schemes is finite flat and of finite presentation.*

Proof. By a limit argument, we may work over a Noetherian base. On every geometric fiber, translation identifies the fibers of f with its kernel, so f is quasi-finite. A quasi-finite morphism between smooth schemes over a field, of the same dimension at corresponding points, is flat by the local flatness criterion for Cohen–Macaulay schemes over regular schemes. The fiberwise criterion of flatness therefore shows that f is flat over S. In the proper case, f is proper and quasi-finite, hence finite; its finite presentation follows from the finite presentation of the smooth group schemes over S. See also [1, Lem. 1.3.1.11]. □

Corollary 2.33. *If $f : A \to A'$ is an isogeny of abelian schemes over S, then $\ker(f)$ is finite flat and of finite presentation, hence finite locally free. Its rank is therefore locally constant on S.*

For an integral variety X, write $K(X)$ for its function field.

Proposition 2.34. *Let $f : A \to A'$ be an isogeny of abelian varieties over a field k. The following are equivalent:*
(1) : *$K(A)$ is a separable extension of $K(A')$;*
(2) : *f is étale;*
(3) : *$\ker(f)$ is an étale group scheme over k.*
The following are also equivalent:
(1) : *$K(A)$ is a purely inseparable extension of $K(A')$;*
(2) : *f is radicial, or equivalently, universally injective;*
(3) : *$\ker(f)$ is geometrically connected.*

Proof. The finite flat morphism f is a torsor under its kernel for the fppf topology. Thus f is étale precisely when its kernel is étale. If the extension of function fields is separable, f is generically étale; after passing to an algebraic closure, translations show that it is étale everywhere. Likewise, a finite group scheme over an algebraically closed field is connected precisely when it has a single geometric point. Since every geometric fiber of f is a translate of the kernel, this is equivalent to universal injectivity, and hence to pure inseparability of the extension of function fields. □

Definition 2.35. An isogeny of abelian varieties is *separable* if it satisfies the first group of equivalent conditions in Proposition 2.34, and *purely inseparable* if it satisfies the second group.

We now consider the multiplication-by-n morphism. For any $n \in \mathbb{Z}$, write

$$[n] : A \longrightarrow A, \qquad A[n] := \ker([n]).$$

If $n = a - b$ with $a, b \geq 0$, the morphism $[n]$ is obtained by adding a copies of the identity and b copies of the inverse:

$$\begin{array}{ccc} A & \xrightarrow{(\mathrm{Id},\ldots,\mathrm{Id},[-1],\ldots,[-1])} & A^{a+b} \\ & \searrow^{[n]} & \downarrow{\scriptstyle m_{a+b}} \\ & & A. \end{array}$$

Here all products are over S, and $[0] = e \circ \pi$. Associativity and the inverse identity

$$\begin{array}{ccc} A & \xrightarrow{(\mathrm{Id}_A,[-1])} & A \times_S A \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle m} \\ S & \xrightarrow[e]{} & A \end{array}$$

show that the construction is independent of the choice of a, b. Since A is commutative, $[n]$ is a homomorphism. Its pullback on invariant differentials and its differential on the Lie algebra are multiplication by n:

$$[n]^* : \underline{\mathrm{Lie}}^\vee_{A/S} \longrightarrow \underline{\mathrm{Lie}}^\vee_{A/S}, \qquad d[n] : \underline{\mathrm{Lie}}_{A/S} \longrightarrow \underline{\mathrm{Lie}}_{A/S}.$$

Proposition 2.36. *For $n \neq 0$, the morphism $[n]$ is an isogeny. On a component of S where A/S has relative dimension g, its degree is n^{2g}. If n is invertible on S, then $[n]$ is finite étale.*

Proof. First suppose n is invertible on S. Its differential is then an isomorphism, so $[n]$ is étale. It is proper and quasi-finite, hence finite. On every geometric fiber its image is both open and closed, and is nonempty; connectedness of the fiber therefore gives surjectivity. The general case and the degree formula will be proved below, after projectivity has been established. □

2.8. **Seesaw.** We use the following form of the seesaw principle.

Proposition 2.37. *Let $\pi : X \to S$ be a proper flat morphism of finite presentation with geometrically integral fibers, where S is locally Noetherian. Then*

$$\mathcal{O}_{S'} \xrightarrow{\sim} (\pi_{S'})_* \mathcal{O}_{X \times_S S'}$$

for every $S' \to S$. Let $\mathcal{L}$ be an invertible sheaf on X. Then

$$S_0 := \{s \in S : \mathcal{L}|_{X_s} \simeq \mathcal{O}_{X_s}\}$$

is closed. There is a unique closed subscheme $S_1 \subset S$, with underlying set S_0, characterized by:
(a) : $\mathcal{L}|_{X \times_S S_1} \simeq \pi_{S_1}^* \mathcal{M}$ for an invertible sheaf $\mathcal{M}$ on S_1;
(b) : a morphism $f : T \to S$ factors through S_1 if and only if

$$\mathcal{L}|_{X \times_S T} \simeq \pi_T^* \mathcal{K}$$

for some invertible sheaf $\mathcal{K}$ on T.

Proof. On a proper geometrically integral fiber with $H^0(X_s, \mathcal{O}_{X_s}) = k(s)$, the sheaf $\mathcal{L}_s$ is trivial if and only if both $H^0(X_s, \mathcal{L}_s)$ and $H^0(X_s, \mathcal{L}_s^{-1})$ are nonzero. Indeed, nonzero sections of these two sheaves have nonzero product, which is a constant, so each is nowhere vanishing. Upper semicontinuity (Theorem 6.4) therefore shows that S_0 is closed.

Fix $s \in S_0$ and work over an affine neighborhood $S = \operatorname{Spec} A$. A finite complex $K^\bullet$ of finite free A-modules, concentrated in nonnegative degrees, computes the cohomology of $\mathcal{L}$ after arbitrary base change. If its first differential is $d : K^0 \to K^1$, put

$$M := \operatorname{coker}\left((K^1)^\vee \xrightarrow{d^\vee} (K^0)^\vee\right).$$

For every A-algebra B,

$$H^0(X_B, \mathcal{L}_B) \simeq \operatorname{Hom}_A(M, B).$$

Since $\dim_{k(s)}(M \otimes_A k(s)) = h^0(X_s, \mathcal{L}_s) = 1$, Nakayama's lemma allows us, after shrinking around s, to write $M \simeq A/I$ for an ideal I. Put $S_1 = \operatorname{Spec}(A/I)$. Then

$$(\pi_{S_1})_* \mathcal{L}_{S_1} \simeq \mathcal{O}_{S_1},$$

and the evaluation morphism

$$\pi_{S_1}^* (\pi_{S_1})_* \mathcal{L}_{S_1} \longrightarrow \mathcal{L}_{S_1}$$

is an isomorphism on X_s. Its failure locus is closed; its image in S_1 is closed by properness. Shrinking the original neighborhood once more makes evaluation an isomorphism everywhere over S_1, proving (a) locally.

To prove (b), work locally on T so that $T = \operatorname{Spec} B$ and $\mathcal{K}$ is trivial. Then

$$\operatorname{Hom}_A(A/I, B) \simeq H^0(X_B, \mathcal{L}_B) \simeq B.$$

The left side is annihilated by I, so $IB = 0$. Thus $T \to S$ factors through S_1. Conversely, (a) gives the required pullback whenever the morphism factors through S_1. This universal property makes the local subschemes glue uniquely, and their underlying set is S_0. □

Remark 2.38. For a proper flat morphism of finite presentation with geometrically connected and geometrically reduced fibers, the natural map

$$\mathcal{O}_{S'} \xrightarrow{\sim} (\pi_{S'})_* \mathcal{O}_{X \times_S S'}$$

is an isomorphism for every base change. In particular, it holds for abelian schemes. Such a morphism is cohomologically flat in degree 0. The geometrically integral hypothesis in the preceding proposition also ensures that nonzero sections of $\mathcal{L}_s$ and $\mathcal{L}_s^{-1}$ have nonzero product.

2.9. **Theorem of the cube.**

Theorem 2.39. *Let k be an algebraically closed field, let X, Y be proper integral k-schemes with points x_0, y_0, and let Z be a connected k-scheme of finite type with a point z_0. If an invertible sheaf $\mathcal{L}$ on $X \times_k Y \times_k Z$ is trivial on*

$$\{x_0\} \times_k Y \times_k Z, \quad X \times_k \{y_0\} \times_k Z, \quad X \times_k Y \times_k \{z_0\},$$

then $\mathcal{L}$ is trivial.

Remark 2.40. The proper integral schemes X, Y have $H^0(X, \mathcal{O}_X) = H^0(Y, \mathcal{O}_Y) = k$, so seesaw applies to their product. We will also use the relative version for abelian schemes over an arbitrary base: a rigidified invertible sheaf on an abelian scheme has a unique compatible cubical structure. This relative statement includes nonreduced bases; see [1, Prop. 3.2.3.1].

Proof. Put $W = X \times_k Y$. By seesaw there is a maximal closed subscheme $Z' \subset Z$ over which $\mathcal{L}$ is pulled back from the base. Its restriction along $(x_0, y_0) \times Z$ is trivial, so $\mathcal{L}|_{W \times Z'}$ is itself trivial. We have $z_0 \in Z'$. It is enough to show that Z' is open, because Z is connected.

Let $z \in Z'$ be a closed point and write locally $Z = \operatorname{Spec} R$, with R Noetherian local and maximal ideal $\mathfrak{m}$, and $Z' = \operatorname{Spec}(R/I)$. Suppose $I \neq 0$. By Krull's intersection theorem, there is an $n \geq 1$ such that

$$I \subset \mathfrak{m}^n, \qquad I \not\subset \mathfrak{m}^{n+1}.$$

Choose an ideal I_2 with $\mathfrak{m}^{n+1} \subset I_2 \subset \mathfrak{m}^n$, such that $\mathfrak{m}^n/I_2 \simeq k$ and $I \not\subset I_2$. Let

$$Z_1 = \operatorname{Spec}(R/\mathfrak{m}^n), \qquad Z_2 = \operatorname{Spec}(R/I_2).$$

The sheaf $\mathcal{L}_1$ is trivial, since $Z_1 \subset Z'$. The square-zero extension

$$0 \longrightarrow k \longrightarrow R/I_2 \longrightarrow R/\mathfrak{m}^n \longrightarrow 0$$

gives, by flatness over Z, an exact sequence

$$0 \longrightarrow \mathcal{O}_W \longrightarrow \mathcal{L}_2 \longrightarrow \mathcal{L}_1 \longrightarrow 0.$$

The obstruction to lifting a trivializing section of $\mathcal{L}_1$ is a class $c \in H^1(W, \mathcal{O}_W)$. By the Künneth formula (Theorem 6.6), restriction to the two axes gives an isomorphism

$$H^1(W, \mathcal{O}_W) \xrightarrow{\sim} H^1(X, \mathcal{O}_X) \oplus H^1(Y, \mathcal{O}_Y).$$

The restrictions of $\mathcal{L}$ to these axes are trivial over all of Z. Moreover, the restriction of any chosen trivializing section of $\mathcal{L}_1$ differs there from a fixed trivialization by a unit of $R/\mathfrak{m}^n$, and such a unit lifts to R/I_2. Hence both restrictions of c vanish. Thus $c = 0$, and a trivializing section lifts to $\mathcal{L}_2$; it remains nowhere vanishing by Nakayama's lemma. This contradicts the maximality of Z', since $I \not\subset I_2$. Therefore $I = 0$.

The ideal of Z' consequently vanishes near each closed point of Z', so Z' is open. It follows that $Z' = Z$, and the trivialization along $(x_0, y_0) \times Z$ removes the pullback from the base. Hence $\mathcal{L}$ is trivial. □

Definition 2.41. Let $\mathcal{L}$ be an invertible sheaf on an abelian scheme $\pi : A \to S$ with identity section e. A *rigidification* of $\mathcal{L}$ is an isomorphism $\mathcal{O}_S \xrightarrow{\sim} e^*\mathcal{L}$. We call $\mathcal{L}$ *rigidified* when such an isomorphism has been chosen.

Remark 2.42. Every invertible sheaf has a canonical rigidified normalization

$$\mathcal{L} \otimes \pi^*(e^*\mathcal{L})^{-1}.$$

Lemma 2.43. *Between two rigidified invertible sheaves on A, there is at most one isomorphism respecting their rigidifications.*

Proof. The ratio of two such isomorphisms is an automorphism of an invertible sheaf, hence multiplication by a global unit on A. Since $\pi_*\mathcal{O}_A = \mathcal{O}_S$, the unit is pulled back from S. Its restriction along e is 1, so the two isomorphisms coincide. □

For a rigidified invertible sheaf $\mathcal{L}$, define on $A \times_S A$

$$\mathcal{D}_2(\mathcal{L}) := m^*\mathcal{L} \otimes \mathrm{pr}_1^*\,\mathcal{L}^{-1} \otimes \mathrm{pr}_2^*\,\mathcal{L}^{-1},$$

and on $A \times_S A \times_S A$ define

$$\begin{aligned}\mathcal{D}_3(\mathcal{L}) := {} & m_{123}^*\mathcal{L} \otimes m_{12}^*\mathcal{L}^{-1} \otimes m_{13}^*\mathcal{L}^{-1} \otimes m_{23}^*\mathcal{L}^{-1} \\ & \otimes \mathrm{pr}_1^*\,\mathcal{L} \otimes \mathrm{pr}_2^*\,\mathcal{L} \otimes \mathrm{pr}_3^*\,\mathcal{L}.\end{aligned}$$

Here m_{ij} and m_{123} add the indicated factors. The rigidification canonically trivializes the restrictions of $\mathcal{D}_3(\mathcal{L})$ to the three coordinate faces on which one factor is e.

Corollary 2.44. *The invertible sheaf $\mathcal{D}_3(\mathcal{L})$ is canonically trivial, compatibly with these face trivializations.*

Corollary 2.45. *For $f, g, h \in A(T)$, where T is an S-scheme, there is an isomorphism*

$$\begin{aligned}(f+g+h)^*\mathcal{L} \simeq {} & (f+g)^*\mathcal{L} \otimes (g+h)^*\mathcal{L} \otimes (h+f)^*\mathcal{L} \\ & \otimes f^*\mathcal{L}^{-1} \otimes g^*\mathcal{L}^{-1} \otimes h^*\mathcal{L}^{-1}.\end{aligned}$$

Proof. Pull back the trivialization of $\mathcal{D}_3(\mathcal{L})$ along $(f, g, h) : T \to A \times_S A \times_S A$. □

Corollary 2.46. *For every $n \in \mathbb{Z}$,*

$$[n]^*\mathcal{L} \simeq \mathcal{L}^{\otimes (n^2+n)/2} \otimes ([-1]^*\mathcal{L})^{\otimes (n^2-n)/2}.$$

Proof. Apply the preceding corollary to $f = [r]$, $g = \mathrm{Id}_A$, and $h = [-1]$. If $F_r = [r]^*\mathcal{L}$, it gives

$$F_{r+1} \otimes F_{r-1} \simeq F_r^{\otimes 2} \otimes \mathcal{L} \otimes [-1]^*\mathcal{L}.$$

Together with $F_0 \simeq \mathcal{O}_A$, $F_1 = \mathcal{L}$, and $F_{-1} = [-1]^*\mathcal{L}$, this determines the displayed formula for all integers n. □

Corollary 2.47 (Theorem of the square)**.** *Let $x, y \in A(S')$ and let $\pi' : A_{S'} \to S'$. Then*

$$T_{x+y}^*\mathcal{L} \otimes \mathcal{L} \simeq T_x^*\mathcal{L} \otimes T_y^*\mathcal{L} \otimes (\pi')^*\big((x+y)^*\mathcal{L} \otimes x^*\mathcal{L}^{-1} \otimes y^*\mathcal{L}^{-1}\big).$$

In particular,

$$[T_{x+y}^*\mathcal{L} \otimes \mathcal{L}] = [T_x^*\mathcal{L} \otimes T_y^*\mathcal{L}]$$

in $\mathrm{Pic}(A_{S'})/(\pi')^*\,\mathrm{Pic}(S')$*. When S' is the spectrum of a field, the last pullback factor is a trivial invertible sheaf.*

Proof. In the preceding three-variable identity take $T = A_{S'}$, $f = \mathrm{Id}_{A_{S'}}$, $g = x \circ \pi'$, and $h = y \circ \pi'$, and rearrange the factors. □

Remark 2.48. For more general commutative group schemes one can still define $\mathcal{D}_3(\mathcal{L})$, but it need not be trivial. A compatible trivialization satisfying the symmetry and cocycle identities is called a *cubical structure*. The relative theorem of the cube gives every rigidified invertible sheaf on an abelian scheme a unique cubical structure. Such structures remain useful when abelian schemes degenerate to more general group schemes; see [1, Secs. 3.2.2–3.2.3].

2.10. **Relative Picard functor.**

Definition 2.49. Let X be a scheme of finite presentation over S, and let $\pi_T : X \times_S T \to T$ be the projection. The relative Picard functor is the contravariant functor

$$\underline{\mathrm{Pic}}(X/S) : (\mathsf{Sch}_S)^{\mathrm{op}} \longrightarrow \mathsf{Ab}, \qquad T \longmapsto \mathrm{Pic}(X \times_S T)/\pi_T^*\,\mathrm{Pic}(T).$$

Here Pic denotes isomorphism classes of invertible sheaves, with tensor product as group law. For general X/S, one takes the associated fppf sheaf when discussing representability. For an abelian scheme the displayed functor is already a sheaf, as follows from its rigidified description below.

For an abelian scheme A/S, define the subfunctor

$$\underline{\mathrm{Pic}}^0(A/S)(T) := \frac{\left\{\begin{array}{l}\text{invertible sheaves } \mathcal{L} \text{ on } A \times_S T \text{ such that} \\ \mathcal{L}_{\bar{t}} \text{ is algebraically equivalent to zero for every geometric } \bar{t} \to T\end{array}\right\}}{\pi_T^* \operatorname{Pic}(T)}.$$

Rigidified normalization gives canonical identifications

$$\underline{\mathrm{Pic}}(A/S) \simeq \underline{\mathrm{Pic}}_e(A/S), \qquad \underline{\mathrm{Pic}}^0(A/S) \simeq \underline{\mathrm{Pic}}^0_e(A/S),$$

where

$$\underline{\mathrm{Pic}}_e(A/S)(T) = \{\text{invertible sheaves on } A \times_S T, \text{ rigidified along } e_T\}/\cong,$$

and the superscript 0 imposes the same fiberwise algebraic-equivalence condition. Isomorphisms on the right respect rigidifications. Tensor product and duals give these functors their commutative group structures.

Remark 2.50. Over an algebraically closed field, two invertible sheaves are algebraically equivalent if their classes can be connected through families of invertible sheaves parametrized by connected algebraic schemes. Algebraic equivalence to zero means equivalence to the trivial invertible sheaf. Over a general field we apply this condition after passing to an algebraic closure.

The theorem of the square gives the following construction.

Theorem 2.51. *For every rigidified invertible sheaf $\mathcal{L}$ on A, the assignment*

$$\begin{aligned}\lambda_{\mathcal{L}}(T) : A(T) &\longrightarrow \underline{\mathrm{Pic}}^0_e(A/S)(T), \\ x &\longmapsto T_x^*\mathcal{L}_T \otimes \mathcal{L}_T^{-1} \otimes \pi_T^*(x^*\mathcal{L}_T)^{-1}\end{aligned}$$

defines a homomorphism of group functors.

Proof. The last factor makes the invertible sheaf canonically rigidified along e_T. Its class on every geometric fiber is algebraically trivial, since the translates of $\mathcal{L}$ form a family parametrized by the connected abelian variety $A_{\bar{t}}$. The square identity above gives additivity; uniqueness of rigidified isomorphisms makes this compatible with base change. □

The functor $\underline{\mathrm{Pic}}^0_e(A/S)$ is represented by an abelian scheme $A^\vee \to S$, called the *dual abelian scheme*; its relative dimension equals that of A. Its universal rigidified invertible sheaf

$$\mathcal{P}_A \quad \text{on } A \times_S A^\vee$$

is the **Poincaré invertible sheaf**. It is trivialized along both $e_A \times_S A^\vee$ and $A \times_S e_{A^\vee}$, with compatible trivializations at the identity. We use this representability theorem as a standard result; see [1, Thm. 1.3.2.3 and Def. 1.3.2.5].

Remark 2.52. In the projective case, representability can be obtained from the Picard-scheme construction. The general theorem does not require a chosen relatively ample invertible sheaf. Below we also discuss the quotient construction arising from such a sheaf.

2.11. **The group $K(\mathcal{L})$.** Let $\mathcal{L}$ be a rigidified invertible sheaf on an abelian scheme $\pi : A \to S$. View $A \times_S A$ as an abelian scheme over its second factor, with identity section (e, Id_A). The sheaf

$$\mathcal{D}_2(\mathcal{L}) = m^*\mathcal{L} \otimes \mathrm{pr}_1^*\,\mathcal{L}^{-1} \otimes \mathrm{pr}_2^*\,\mathcal{L}^{-1}$$

is rigidified along this section. Seesaw gives a maximal closed subscheme $K(\mathcal{L}) \subset A$ on which this family is trivial. Equivalently,

$$K(\mathcal{L}) = \ker\left(\lambda_{\mathcal{L}} : A \longrightarrow A^\vee\right),$$

so it is a closed subgroup scheme. Explicitly, a point $x \in A(T)$ belongs to $K(\mathcal{L})(T)$ precisely when

$$T_x^*\mathcal{L}_T \otimes \mathcal{L}_T^{-1} \otimes \pi_T^*(x^*\mathcal{L}_T)^{-1}$$

is trivial as a rigidified invertible sheaf.

Theorem 2.53. *Let A be an abelian variety over a field k, let D be an effective Cartier divisor on A, and put $\mathcal{L} = \mathcal{O}_A(D)$. The following are equivalent:*

(1) : *The subgroup*

$$H = \{x \in A(\overline{k}) : T_x^* D = D\}$$

is finite. This is equality of divisors, not merely of divisor classes.

(2) : *$K(\mathcal{L})$ is finite.*

(3) : *The linear system*

$$|2D| = \{D' \geq 0 : D' \sim 2D\}$$

is base point free and its associated morphism $A \to \mathbb{P}_k^N$ is finite.

(4) : *$\mathcal{L}$ is ample.*

Proof. These properties can be checked after extending k to $\overline{k}$, so assume k is algebraically closed.

(2) $\Rightarrow$ (1) : Clearly $H \subset K(\mathcal{L})(k)$.

(4) $\Rightarrow$ (2) : Let $B = (K(\mathcal{L})_{\mathrm{red}})^0$. Over k this is an abelian subvariety of A, and $\mathcal{L}_B := \mathcal{L}|_B$ is ample. By the defining property of $K(\mathcal{L})$, the sheaf $\mathcal{D}_2(\mathcal{L}_B)$ is trivial. Pulling its inverse back along $(\mathrm{Id}_B, [-1]_B)$ gives

$$\mathcal{L}_B \otimes [-1]_B^* \mathcal{L}_B \simeq \mathcal{O}_B.$$

The left side is ample. A positive-dimensional proper scheme cannot have its trivial invertible sheaf ample, so $\dim B = 0$. Thus $K(\mathcal{L})$ is zero-dimensional and proper, hence finite.

(3) $\Rightarrow$ (4) : If $\phi : A \to \mathbb{P}^N$ is the finite morphism defined by $|2D|$, then $\mathcal{L}^{\otimes 2} \simeq \phi^* \mathcal{O}_{\mathbb{P}^N}(1)$ is ample. Therefore $\mathcal{L}$ is ample.

(1) $\Rightarrow$ (3) : The theorem of the square gives

$$\mathcal{L}^{\otimes 2} \simeq T_x^* \mathcal{L} \otimes T_{-x}^* \mathcal{L}, \qquad x \in A(k).$$

Hence $T_x^* D + T_{-x}^* D \in |2D|$. For a fixed $u \in A(k)$, choose x outside the two proper closed sets defined by $u + x \in \mathrm{Supp}(D)$ and $u - x \in \mathrm{Supp}(D)$. The resulting divisor avoids u. Thus $|2D|$ is base point free and defines a morphism $\phi : A \to \mathbb{P}^N$.

If ϕ were not finite, a positive-dimensional proper fiber would contain an integral curve C. Every member of $|2D|$ would then either contain C or be disjoint from it. In particular, for a general x the divisor $T_x^* D + T_{-x}^* D$ is disjoint from C: it is enough to choose it to avoid one point of C.

Claim 2.54. *If F is a prime divisor disjoint from C, then*

$$T_{c_1 - c_2}^* F = F, \qquad c_1, c_2 \in C(k).$$

Proof of claim. Translates of $\mathcal{O}_A(F)$ are algebraically equivalent, so their restrictions to the normalization of C have the same degree, namely 0. Consequently, any translate $T_a(C)$ either lies in F or is disjoint from F. Given $y \in F(k)$, the curve $T_{y-c_2}(C)$ meets F at y, so it is contained in F. Thus $y + c_1 - c_2 \in F$. Translation by $c_1 - c_2$ preserves the irreducible divisor F, proving the claim. □

Write $D = \sum_i n_i D_i$. Apply the claim to every prime component of a translate of D disjoint from C. Translating back shows that every $c_1 - c_2$ preserves every D_i, hence preserves D. Fixing c_2 gives infinitely many such elements as c_1 varies on C, contradicting (1). Therefore ϕ is finite. □

Lemma 2.55. *Let X be a separated Noetherian normal integral scheme and let $U \subset X$ be a nonempty affine open subset. Then every irreducible component of $X \setminus U$ has codimension 1 in X.*

Proof. Let s be the generic point of such a component. Since the intersection of two affine opens in a separated scheme is affine, we may localize at s. We obtain $X = \operatorname{Spec} R$, with R a Noetherian normal local domain, and $U = X \setminus \{\mathfrak{m}\}$ affine. If $\dim R \geq 2$, normality gives

$$R = \bigcap_{\mathrm{ht}\, \mathfrak{p} = 1} R_{\mathfrak{p}}$$

inside its fraction field. Since U contains every height-one point, every regular function on U lies in this intersection. Hence $\Gamma(U, \mathcal{O}_U) = R$, and the affine open immersion $U \hookrightarrow X$ would be an isomorphism, a contradiction. Therefore $\dim R = 1$, as required. □

Corollary 2.56. *Every abelian variety over a field is projective.*

Proof. First let $k = \overline{k}$. Take an affine neighborhood U of $e \in A$. The complement has pure codimension 1; write its prime components as $D_1, \dots, D_r$ and set $D = \sum_i D_i$. Since A is smooth, D is a Cartier divisor. Its stabilizer H is a closed subgroup of A and preserves U. Because $e \in U$, we have $H \subset U$. Thus H is both affine and proper, hence finite. The preceding theorem shows that $\mathcal{O}_A(D)$ is ample, so A is projective. For general k, an ample invertible sheaf over $\overline{k}$ is defined over a finite extension of k; its norm gives an ample invertible sheaf over k. Hence the result descends to k. □

We can now finish the proof for multiplication by any nonzero integer.

Proposition 2.57. *The morphism $[n] : A \to A$ is an isogeny for every $n \neq 0$.*

Proof. It suffices first to work on a geometric fiber $A_{\bar{s}}$. By Corollary 2.56, choose an ample invertible sheaf $\mathcal{L}$ on this fiber and put

$$\mathcal{M} = \mathcal{L} \otimes [-1]^*\mathcal{L}.$$

Then $\mathcal{M}$ is symmetric and ample. The multiplication formula gives

$$[n]^*\mathcal{M} \simeq \mathcal{M}^{\otimes n^2},$$

which is ample for $n \neq 0$. Its restriction to any fiber of $[n]$ is trivial, so that fiber cannot have positive dimension. Thus $[n]$ is quasi-finite on every geometric fiber, hence quasi-finite globally. Properness makes it finite. Its image on each geometric fiber has dimension $\dim A_{\bar{s}}$, and is closed, so it is the whole fiber. Thus $[n]$ is surjective and has finite kernel, and the isogeny lemma gives flatness.

On a fiber of dimension g, the intersection formula for a finite surjective morphism gives

$$\deg([n])\,(c_1(\mathcal{M})^g) = (c_1([n]^*\mathcal{M})^g) = n^{2g}(c_1(\mathcal{M})^g).$$

The last intersection number is positive, so $\deg([n]) = n^{2g}$. Since degree is locally constant in a finite flat family, this proves the asserted relative degree formula as well. □

Corollary 2.58. *For $n \neq 0$, the morphism $[n]$ is finite flat and surjective, and $A[n]$ is finite locally free of rank n^{2g} on a component where A/S has relative dimension g.*

In particular, $\deg([n]) = \operatorname{rank}_{\mathcal{O}_S} A[n]$.

2.12. $\underline{\mathrm{Pic}}^0_e(A/S)$ **revisited.** Fix an abelian scheme $\pi : A \to S$ with identity section e.

Lemma 2.59. *Let A be an abelian variety over an algebraically closed field k and let $\mathcal{L}$ be rigidified. Then*

$$\begin{aligned} \mathcal{L} \in \mathrm{Pic}^0(A) &\iff T_x^*\mathcal{L} \simeq \mathcal{L} \quad \text{for every } x \in A(k) \\ &\iff m^*\mathcal{L} \simeq \mathrm{pr}_1^*\,\mathcal{L} \otimes \mathrm{pr}_2^*\,\mathcal{L}. \end{aligned}$$

In relative form, $\mathcal{L} \in \underline{\mathrm{Pic}}^0_e(A/S)(S)$ if and only if $\lambda_{\mathcal{L}} = 0$, equivalently $\mathcal{D}_2(\mathcal{L})$ is trivial as a rigidified invertible sheaf.

Proof. The first equivalence is the translation criterion for algebraic equivalence on an abelian variety. One direction can also be seen directly from rigidity: the homomorphism $\lambda_{\mathcal{L}}$ is constant in a connected family of algebraically equivalent invertible sheaves, and equals zero for the trivial sheaf. The converse is the usual characterization of the identity component of the Picard scheme by translation-invariant invertible sheaves. We use this standard Picard-theory result, together with its relative form; see [1, Sec. 1.3.2].

For the second equivalence, restrict $\mathcal{D}_2(\mathcal{L})$ to $A \times \{x\}$ and to $\{e\} \times A$. The first restriction is the rigidified normalization of $T_x^*\mathcal{L} \otimes \mathcal{L}^{-1}$, while the second is canonically trivial. Seesaw therefore gives the equivalence. Relatively, $\mathcal{D}_2(\mathcal{L}) \simeq (\mathrm{Id}_A \times \lambda_{\mathcal{L}})^*\mathcal{P}_A$, so its triviality is precisely the vanishing of $\lambda_{\mathcal{L}}$. □

Lemma 2.60. *For rigidified invertible sheaves $\mathcal{L}, \mathcal{M}$ on A,*

$$\lambda_{\mathcal{L}\otimes\mathcal{M}} = \lambda_{\mathcal{L}} + \lambda_{\mathcal{M}}, \qquad \lambda_{\mathcal{L}^{-1}} = -\lambda_{\mathcal{L}}.$$

Proof. Both identities follow by expanding the definition of $\mathcal{D}_2$ and using the universal property of the Poincaré invertible sheaf. □

Lemma 2.61. *If $\mathcal{L} \in \underline{\mathrm{Pic}}_e(A/S)(S)$ and $\lambda_{\mathcal{L}} = 0$, then*

$$[n]^*\mathcal{L} \simeq \mathcal{L}^{\otimes n}, \qquad n \in \mathbb{Z}.$$

Proof. Pull back the identity $m^*\mathcal{L} \simeq \mathrm{pr}_1^*\mathcal{L} \otimes \mathrm{pr}_2^*\mathcal{L}$ along repeated addition. Pulling it back along $(\mathrm{Id}_A, [-1])$ gives $[-1]^*\mathcal{L} \simeq \mathcal{L}^{-1}$, which handles negative n as well. □

Lemma 2.62. *Let A be an abelian variety over a field k. If $\lambda_{\mathcal{L}} = 0$ and $\mathcal{L}$ is nontrivial, then $H^i(A, \mathcal{L}) = 0$ for every $i \geq 0$. Consequently, for an abelian scheme $\pi : A \to S$, if $\lambda_{\mathcal{L}} = 0$ and $\mathcal{L}_{\bar{s}}$ is nontrivial on every geometric fiber, then $R^i\pi_*\mathcal{L} = 0$ for every $i \geq 0$.*

Proof. For the field statement, we may extend scalars to an algebraic closure: cohomology commutes with field extension, and an invertible sheaf on A which becomes trivial after field extension was already trivial, by its rigidified normalization and descent.

If $H^0(A, \mathcal{L}) \neq 0$, write $\mathcal{L} \simeq \mathcal{O}_A(D)$ with $D \geq 0$. Lemma 2.61 gives

$$\mathcal{O}_A(D + [-1]^*D) \simeq \mathcal{L} \otimes [-1]^*\mathcal{L} \simeq \mathcal{O}_A.$$

An effective principal divisor on a proper integral variety is zero. Hence $D = 0$, contrary to nontriviality. Thus $H^0(A, \mathcal{L}) = 0$.

Suppose inductively that $H^i(A, \mathcal{L}) = 0$ for $i < l$. By Lemma 2.59 and the Künneth formula,

$$\begin{aligned} H^l(A \times_k A, m^*\mathcal{L}) &\simeq \bigoplus_{i+j=l} H^i(A, \mathcal{L}) \otimes_k H^j(A, \mathcal{L}) \\ &= 0. \end{aligned}$$

The section $\eta : A \to A \times_k A$, $a \mapsto (a, e)$, satisfies $m \circ \eta = \mathrm{Id}_A$, so $\eta^* \circ m^*$ is the identity on cohomology. Thus m^* is injective and $H^l(A, \mathcal{L}) = 0$. This finishes the induction.

In a family, the hypothesis gives vanishing on all geometric fibers. Cohomology and base change, or a local finite complex computing derived pushforward, then gives $R^i\pi_*\mathcal{L} = 0$. Nontriviality of the invertible sheaf on the total space alone would not suffice for this relative conclusion. □

Theorem 2.63. *Let A be an abelian variety over an algebraically closed field k. If $K(\mathcal{L})$ is finite and $\lambda_{\mathcal{M}} = 0$, then there exists $x \in A(k)$ such that*

$$\mathcal{M} \simeq T_x^*\mathcal{L} \otimes \mathcal{L}^{-1}.$$

Proof. The homomorphism $\lambda_{\mathcal{L}} : A \to A^\vee$ has finite kernel. Its image is closed, since A is proper, and has dimension $\dim A = \dim A^\vee$. Thus it is surjective. By Lemma 2.59, $\mathcal{M}$ represents a point of $A^\vee(k)$, so it lies in the image. Over a field, the additional rigidifying pullback factor is a trivial invertible sheaf, giving the displayed isomorphism. □

Corollary 2.64. *Over an algebraically closed field k, finiteness of $K(\mathcal{L})$ implies that*

$$\lambda_{\mathcal{L}}(k) : A(k) \longrightarrow A^\vee(k)$$

is surjective.

Corollary 2.65. *If $\mathcal{L}$ is relatively ample on an abelian scheme A/S, then $\lambda_{\mathcal{L}} : A \to A^\vee$ is an isogeny and $K(\mathcal{L})$ is finite locally free over S.*

Proof. On every geometric fiber, a positive power of $\mathcal{L}$ has an effective divisor and the ampleness theorem above shows that its K is finite. Since $K(\mathcal{L}) \subset K(\mathcal{L}^{\otimes r})$, its kernel is also finite on each fiber. The preceding theorem gives fiberwise surjectivity. Thus $\lambda_{\mathcal{L}}$ is an isogeny of abelian schemes, and the isogeny lemma gives the conclusion. □

2.13. **Theory of descent.** We use the representability theorem for the relative Picard functor: for an abelian scheme A/S, the functor $\underline{\mathrm{Pic}}^0_e(A/S)$ is represented by an abelian scheme $A^\vee/S$, called the *dual abelian scheme*. Its universal rigidified invertible sheaf on $A\times_S A^\vee$ is denoted by $\mathscr{P}_A$. This is [1, Thm. 1.3.2.3 and Def. 1.3.2.5]. We explain how a relatively ample invertible sheaf realizes $A^\vee$ as a quotient, and how descent recovers $\mathscr{P}_A$.

Let $\mathcal{L}$ be an invertible sheaf on A, rigidified along the identity section. The homomorphism of functors $\lambda_{\mathcal{L}}$ gives a homomorphism of schemes

$$\lambda_{\mathcal{L}} : A \longrightarrow A^\vee$$

by Yoneda's lemma, with kernel $K(\mathcal{L})$. Suppose that $\mathcal{L}$ is relatively ample over S. Then $K(\mathcal{L})$ is finite locally free over S, and $\lambda_{\mathcal{L}}$ is an isogeny. In particular, it is finite locally free and surjective.

Work locally on S, say $S = \operatorname{Spec} R$. Put $K = K(\mathcal{L})$. The fppf quotient $\overline{A} := A/K$ is represented by a scheme, and the quotient morphism $q : A \to \overline{A}$ is a K-torsor. Thus the following square is Cartesian:

$$\begin{array}{ccc} A\times_S K & \xrightarrow{m} & A \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle q} \\ A & \xrightarrow[q]{} & \overline{A}. \end{array}$$

Here $m(x,a) = x + a$, and the corresponding isomorphism is

$$A\times_S K \xrightarrow{\sim} A\times_{\overline{A}} A, \qquad (x,a)\longmapsto (x, x+a).$$

Since K/S is finite locally free, so is q. The quotient exists by the standard quotient theorem for a finite locally free group acting on a projective scheme. Affine locally it is described by invariants, as follows.

For an affine open $\overline{U}\subset\overline{A}$, its inverse image $U = q^{-1}(\overline{U})$ is affine and stable under K. Write

$$B = \mathcal{O}_A(U), \qquad C = \mathcal{O}_K(K).$$

The action and the projection induce two homomorphisms

$$m^*, \mathrm{pr}_1^* : B \rightrightarrows B\otimes_R C,$$

and faithfully flat descent gives the equalizer diagram

$$0 \longrightarrow \mathcal{O}_{\overline{A}}(\overline{U}) \longrightarrow B \underset{\mathrm{pr}_1^*}{\overset{m^*}{\rightrightarrows}} B\otimes_R C.$$

Conversely, the invariant rings on invariant affine neighborhoods give this local description of the quotient; the quotient theorem ensures that these local constructions glue.

Because $\lambda_{\mathcal{L}}$ kills K, it factors through

$$A \xrightarrow{q} \overline{A} \xrightarrow{\overline{\lambda}_{\mathcal{L}}} A^\vee.$$

The second map is a monomorphism of fppf sheaves: if two sections have the same image, lift them fppf locally to A; their difference lies in K, so their images in A/K coincide. It is also surjective as an fppf sheaf because $\lambda_{\mathcal{L}}$ is fppf surjective. Consequently,

$$\overline{\lambda}_{\mathcal{L}} : A/K(\mathcal{L}) \xrightarrow{\sim} A^\vee.$$

In particular, $\overline{A}/S$ is an abelian scheme. We also recall the following closed immersion criterion.

Proposition 2.66 (Closed immersion criterion)**.** *A proper monomorphism of schemes is a closed immersion. In particular, a proper, unramified, universally injective morphism is a closed immersion.*

The quotient construction also makes sense before identifying $\overline{A}$ with $A^\vee$. Its group law descends from that of A, and it is proper over S because A is proper and q is finite surjective. Flatness over S descends along the faithfully flat map q. On each geometric fiber, the quotient of an abelian variety by a finite subgroup scheme is again an abelian variety. Hence the fibers are smooth and connected, and $\overline{A}/S$ is an abelian scheme.

Upshot 2.67. The quotient $A/K(\mathcal{L})$ realizes the dual abelian scheme once the representability theorem is known. To recover the universal object, we descend $\mathcal{D}_2(\mathcal{L})$ along the second factor of $A \times_S A$. The existence of this descent is a concrete consequence of the theorem of the cube. Representability itself is the theorem cited above.

Write $\mathcal{B} := \mathcal{D}_2(\mathcal{L})$. The universal property gives a canonical isomorphism

$$\mathcal{B} \simeq (\mathrm{Id}_A \times \lambda_{\mathcal{L}})^* \mathscr{P}_A.$$

In particular, K is the maximal closed subscheme of A on which the second parameter of $\mathcal{B}$ is trivial: the restriction $\mathcal{B}|_{A\times_S K}$ has a unique trivialization compatible with the rigidification along $e \times_S K$.

The descent diagram for q is

$$A \times_{\overline{A}} A \times_{\overline{A}} A \;\begin{array}{c}\xrightarrow{\mathrm{pr}_{12}}\\ \xrightarrow{\mathrm{pr}_{13}}\\ \xrightarrow{\mathrm{pr}_{23}}\end{array}\; A \times_{\overline{A}} A \;\begin{array}{c}\xrightarrow{\mathrm{pr}_1}\\ \xrightarrow[\mathrm{pr}_2]{}\end{array}\; A \xrightarrow{q} \overline{A}.$$

Under the identifications

$$\begin{aligned} A \times_S K &\xrightarrow{\sim} A \times_{\overline{A}} A, & (x,a) &\longmapsto (x, x+a),\\ A \times_S K \times_S K &\xrightarrow{\sim} A \times_{\overline{A}} A \times_{\overline{A}} A, & (x,a,b) &\longmapsto (x, x+a, x+a+b), \end{aligned}$$

the three maps on the left become

$$(x,a,b) \longmapsto (x,a), \qquad (x,a,b) \longmapsto (x,a+b), \qquad (x,a,b) \longmapsto (x+a,b),$$

and the two maps on the right become $(x,a) \mapsto x$ and $(x,a) \mapsto x+a$.

Apply this diagram after taking its product with the first factor A. To descend $\mathcal{B}$ along

$$\mathrm{Id}_A \times q : A \times_S A \longrightarrow A \times_S \overline{A},$$

we need an isomorphism over $A \times_S A \times_S K$

$$\phi : (\mathrm{Id}_A \times m)^* \mathcal{B} \xrightarrow{\sim} (\mathrm{Id}_A \times \mathrm{pr}_1)^* \mathcal{B}$$

satisfying the cocycle condition. In functorial points, write

$$\phi_{u,x,a} : \mathcal{B}_{u,x+a} \xrightarrow{\sim} \mathcal{B}_{u,x}.$$

The required condition is

$$\phi_{u,x,a+b} = \phi_{u,x,a} \circ \phi_{u,x+a,b}$$

for every S-scheme T, $u, x \in A(T)$, and $a, b \in K(T)$.

The theorem of the cube supplies a canonical trivialization

$$\mathcal{D}_3(\mathcal{L}) \simeq \mathcal{O}_{A\times_S A\times_S A}.$$

Equivalently, it supplies a canonical isomorphism

$$\psi : (\mathrm{Id}_A \times m)^* \mathcal{B} \xrightarrow{\sim} (\mathrm{Id}_A \times \mathrm{pr}_1)^* \mathcal{B} \otimes (\mathrm{Id}_A \times \mathrm{pr}_2)^* \mathcal{B}.$$

Thus

$$\psi_{u,x,y} : \mathcal{B}_{u,x+y} \xrightarrow{\sim} \mathcal{B}_{u,x} \otimes \mathcal{B}_{u,y}.$$

The rigidifications uniquely determine this isomorphism and imply its associativity:

$$\begin{array}{ccc} \mathcal{B}_{u,x+y+z} & \xrightarrow{\psi_{u,x+y,z}} & \mathcal{B}_{u,x+y} \otimes \mathcal{B}_{u,z} \\ {\scriptstyle \psi_{u,x,y+z}}\downarrow & & \downarrow{\scriptstyle \psi_{u,x,y}\otimes \mathrm{Id}} \\ \mathcal{B}_{u,x} \otimes \mathcal{B}_{u,y+z} & \xrightarrow[\mathrm{Id}\otimes\psi_{u,y,z}]{} & \mathcal{B}_{u,x} \otimes \mathcal{B}_{u,y} \otimes \mathcal{B}_{u,z}. \end{array}$$

Here, as usual, the fiber notation means pullback of an invertible sheaf to T.

Let $\epsilon_{u,a} : \mathcal{B}_{u,a} \xrightarrow{\sim} \mathcal{O}_T$ be the canonical trivialization for $a \in K(T)$. Define

$$\phi_{u,x,a} : \mathcal{B}_{u,x+a} \xrightarrow{\psi_{u,x,a}} \mathcal{B}_{u,x} \otimes \mathcal{B}_{u,a} \xrightarrow{\mathrm{Id} \otimes \epsilon_{u,a}} \mathcal{B}_{u,x}.$$

The compatibility of ϵ with ψ follows from uniqueness of the rigidified trivialization. Associativity then gives the commutative diagram

$$\begin{array}{ccc} \mathcal{B}_{u,x+a+b} & \xrightarrow{\phi_{u,x+a,b}} & \mathcal{B}_{u,x+a} \\ {\scriptstyle \phi_{u,x,a+b}}\downarrow & & \downarrow{\scriptstyle \phi_{u,x,a}} \\ \mathcal{B}_{u,x} & \xrightarrow[\mathrm{Id}]{} & \mathcal{B}_{u,x}. \end{array}$$

Consequently, faithfully flat descent produces an invertible sheaf $\mathscr{P}$ on $A \times_S \overline{A}$ with

$$(\mathrm{Id}_A \times q)^* \mathscr{P} \simeq \mathcal{B}.$$

Its rigidifications descend as well. Under $\overline{A} \simeq A^\vee$, it is the Poincaré invertible sheaf.

Theorem 2.68. *The abelian scheme $\overline{A}$ represents $\underline{\mathrm{Pic}}^0_e(A/S)$, with universal object $\mathscr{P}$. Namely, for every S-scheme T and every rigidified invertible sheaf $\mathcal{M} \in \underline{\mathrm{Pic}}^0_e(A/S)(T)$, there is a unique S-morphism $\alpha : T \to \overline{A}$ and a unique rigidification-preserving isomorphism*

$$\mathcal{M} \simeq (\mathrm{Id}_A \times \alpha)^* \mathscr{P}.$$

Proof. The representability theorem gives a unique morphism $\beta : T \to A^\vee$ classifying $\mathcal{M}$. Set $\alpha = \overline{\lambda}_{\mathcal{L}}^{-1} \circ \beta$. The descended sheaf satisfies

$$\mathscr{P} \simeq (\mathrm{Id}_A \times \overline{\lambda}_{\mathcal{L}})^* \mathscr{P}_A :$$

both sides pull back to $\mathcal{B}$ with the same rigidified descent datum. This proves the asserted isomorphism. Uniqueness follows from representability and from the fact that a rigidified invertible sheaf on an abelian scheme has no nontrivial rigidification-preserving automorphism, since $\pi_* \mathcal{O}_A = \mathcal{O}_S$ universally.

One may express the same argument using the graph of α. On $A \times_S T \times_S \overline{A}$, put

$$\widetilde{\mathcal{M}} := \mathrm{pr}_{12}^* \mathcal{M}^\vee \otimes \mathrm{pr}_{13}^* \mathscr{P}.$$

The subfunctor of $T \times_S \overline{A}$ on which this rigidified sheaf is trivial is represented by the graph Γ_α. Its projection to T is therefore an isomorphism. □

2.14. **The Weil pairing.** Let $f : A \to B$ be an isogeny over S. Pullback of rigidified invertible sheaves defines its dual isogeny

$$f^\vee : B^\vee \longrightarrow A^\vee.$$

The universal properties of the Poincaré sheaves give a canonical isomorphism

$$(f \times \mathrm{Id}_{B^\vee})^* \mathscr{P}_B \simeq (\mathrm{Id}_A \times f^\vee)^* \mathscr{P}_A$$

on $A \times_S B^\vee$; see [1, Lem. 1.3.2.10 and Sec. 5.2.4]. Put

$$K := \ker f, \qquad K^\vee := \ker f^\vee.$$

We will identify $K^\vee$ with the Cartier dual

$$\underline{\mathrm{Hom}}_S(K, \mathbb{G}_{m,S})$$

and obtain a canonical perfect pairing

$$e_K : K \times_S K^\vee \longrightarrow \mathbb{G}_{m,S}.$$

The underline is essential: this is the group functor of characters after arbitrary base change, rather than just the group of homomorphisms over S.

We work with functorial points; all constructions below commute with base change, so we write them over S. For $h \in K^\vee(S)$, let

$$\mathcal{N}_h := (\mathrm{Id}_B \times h)^* \mathscr{P}_B.$$

This is a rigidified invertible sheaf on B, with

$$e_B^*\mathcal{N}_h \simeq \mathcal{O}_S.$$

Since $f^\vee(h) = e_{A^\vee}$, the preceding comparison gives a canonical rigidified trivialization

$$\begin{aligned} f^*\mathcal{N}_h &\simeq (f \times \mathrm{Id}_{B^\vee})^* \mathscr{P}_B|_{A\times_S\{h\}} \\ &\simeq (\mathrm{Id}_A \times f^\vee)^* \mathscr{P}_A\big|_{A\times_S\{h\}} \\ &\simeq \mathscr{P}_A|_{A\times_S\{e_{A^\vee}\}} \simeq \mathcal{O}_A. \end{aligned}$$

Identify B with A/K. By descent, $\mathcal{N}_h$ is described by a K-equivariant structure on $\mathcal{O}_A$. For each functorial point $a \in K$, this gives an isomorphism

$$\kappa_h(a) : T_a^*\mathcal{O}_A \xrightarrow{\sim} \mathcal{O}_A.$$

Translation also gives its canonical structural isomorphism

$$\mathrm{str}(a) : T_a^*\mathcal{O}_A \xrightarrow{\sim} \mathcal{O}_A.$$

Their ratio is multiplication by a unit pulled back from the base, because $\pi_*\mathcal{O}_A = \mathcal{O}_S$ universally. With Lan's sign convention, define

$$\widetilde{h}(a) := (e^*\,\mathrm{str}(a)) \circ (e^*\kappa_h(a))^{-1} \in \mathbb{G}_{m,S}.$$

Thus $\widetilde{h} : K \to \mathbb{G}_{m,S}$ is a morphism.

For completeness, write $\mathcal{O}_A(a) := a^*\mathcal{O}_A$ and use $\mathrm{str}(a)$ and $\kappa_h(a)$ also for their restrictions along e. The structural isomorphisms satisfy

$$\begin{array}{ccc} \mathcal{O}_A(a+a') & \xrightarrow{\text{can.}} & \mathcal{O}_A(a) \otimes \mathcal{O}_A(a') \\ {\scriptstyle \mathrm{str}(a+a')}\downarrow & & \downarrow{\scriptstyle \mathrm{str}(a)\otimes\mathrm{str}(a')} \\ \mathcal{O}_S & \xrightarrow[\text{can.}]{} & \mathcal{O}_S \otimes \mathcal{O}_S. \end{array}$$

The cocycle condition on the descent datum gives the analogous diagram

$$\begin{array}{ccc} \mathcal{O}_A(a+a') & \xrightarrow{\text{can.}} & \mathcal{O}_A(a) \otimes \mathcal{O}_A(a') \\ {\scriptstyle \kappa_h(a+a')}\downarrow & & \downarrow{\scriptstyle \kappa_h(a)\otimes\kappa_h(a')} \\ \mathcal{O}_S & \xrightarrow[\text{can.}]{} & \mathcal{O}_S \otimes \mathcal{O}_S. \end{array}$$

All tensor products in these diagrams are over $\mathcal{O}_S$. Comparing them gives

$$\widetilde{h}(a+a') = \widetilde{h}(a)\widetilde{h}(a'),$$

so $\widetilde{h}$ is a character of K.

Conversely, a character of K gives a descent datum on $\mathcal{O}_A$ and hence a rigidified invertible sheaf on B whose pullback to A is trivial. It is fiberwise algebraically equivalent to zero: its image in the Néron–Severi group is killed by pullback along an isogeny, and this pullback is injective. Thus it defines a point of $\ker f^\vee$. These constructions are inverse, giving

$$K^\vee \xrightarrow{\sim} \underline{\mathrm{Hom}}_S(K, \mathbb{G}_{m,S}), \qquad h \longmapsto \widetilde{h}.$$

In particular, the evaluation pairing

$$e_K(a,h) := \widetilde{h}(a)$$

is perfect, and f and $f^\vee$ have the same degree.

For $f = [n] : A \to A$, $n \geq 1$, the identity

$$[n]^*\mathcal{L} \simeq \mathcal{L}^{\otimes n} \qquad (\mathcal{L} \in \underline{\mathrm{Pic}}^0_e(A/S))$$

shows that $f^\vee = [n]$ on $A^\vee$. We obtain a perfect pairing

$$e_{A[n]} : A[n] \times_S A^\vee[n] \longrightarrow \mathbb{G}_{m,S}.$$

Since it is bilinear,

$$e_{A[n]}(x,y)^n = e_{A[n]}(nx, y) = 1.$$

It therefore factors through the **Weil pairing**

$$e_{A[n]} : A[n] \times_S A^\vee[n] \longrightarrow \mu_{n,S}.$$

This remains a perfect pairing of finite locally free group schemes even when n is not invertible on S.

For a homomorphism $\lambda : A \to A^\vee$, define the **λ-Weil pairing** by

$$e^\lambda : A[n] \times_S A[n] \longrightarrow \mu_{n,S}, \qquad e^\lambda(x,y) := e_{A[n]}(x, \lambda(y)).$$

If λ is a polarization, this pairing is alternating. If, moreover, $\deg \lambda$ is relatively prime to n, it is perfect, since $\lambda : A[n] \to A^\vee[n]$ is then an isomorphism.

The general pairing may also be described by the following diagram. For $a \in K(T)$ and $h \in K^\vee(T)$, set

$$\mathcal{Q}_{a,h} := (f \times \mathrm{Id}_{B^\vee})^* \mathscr{P}_B|_{(a,h)}\,, \quad \mathcal{R}_{a,h} := (\mathrm{Id}_A \times f^\vee)^* \mathscr{P}_A\big|_{(a,h)}\,.$$

Then

$$\begin{array}{ccccc} \mathcal{Q}_{a,h} & \xrightarrow{\sim} & \mathscr{P}_B|_{(f(a),h)} & \xrightarrow{\text{rig.}} & \mathcal{O}_T \\ {\scriptstyle\sim}\downarrow & & & & \downarrow{\scriptstyle e_K(a,h)} \\ \mathcal{R}_{a,h} & \xrightarrow[\sim]{} & \mathscr{P}_A|_{(a,f^\vee(h))} & \xrightarrow[\text{rig.}]{} & \mathcal{O}_T \end{array}$$

commutes, where $f(a) = e_B$ and $f^\vee(h) = e_{A^\vee}$.

Upshot 2.69. There are two natural trivializations obtained from the two rigidifications of the Poincaré sheaves. Their ratio is the Weil pairing $e_K(a,h)$.

2.15. **Siegel Moduli Problem.** We give a brief definition of the *Siegel moduli problem*, which will reappear in the deformation theory of abelian schemes. It is the case without additional endomorphisms of the moduli problems in [1, Secs. 1.3.6 and 1.4.1].

Definition 2.70. For an integer $g \geq 1$, the standard symplectic pairing on $\mathbb{Z}^{2g} = \mathbb{Z}^g \oplus \mathbb{Z}^g$ is

$$\langle (x,y), (x',y') \rangle := x^{\mathsf{t}} y' - y^{\mathsf{t}} x'.$$

Its matrix in the standard basis is

$$J = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}.$$

For $n \geq 1$, it induces a perfect alternating pairing on $(\mathbb{Z}/n\mathbb{Z})^{2g}$.

A polarization of A/S is a homomorphism $\lambda : A \to A^\vee$ that, étale locally on S, is $\lambda_{\mathcal{L}}$ for a relatively ample invertible sheaf $\mathcal{L}$. It is *principal* if it is an isomorphism.

Definition 2.71. Fix $g \geq 1$, $n \geq 3$, and a primitive nth root of unity ζ_n. Over $\operatorname{Spec} \mathbb{Z}[1/n, \zeta_n]$, the Siegel moduli functor $\mathcal{A}_{g,n}$ sends S to the set of isomorphism classes of triples (A, λ, α_n), where

(1) : $A \to S$ is an abelian scheme of relative dimension g;

(2) : $\lambda : A \to A^\vee$ is a principal polarization;

(3) : $\alpha_n : (\mathbb{Z}/n\mathbb{Z})^{2g}_S \xrightarrow{\sim} A[n]$ is an isomorphism satisfying

$$e^\lambda(\alpha_n(v), \alpha_n(w)) = \zeta_n^{\langle v,w \rangle}.$$

An isomorphism of triples is an isomorphism $f : A \to A'$ with $f^\vee \lambda' f = \lambda$ and $f \circ \alpha_n = \alpha'_n$.

For $n \geq 3$, the level structure eliminates automorphisms, and this functor is represented by a smooth quasi-projective scheme of relative dimension $g(g+1)/2$. Without this level structure, the natural moduli object is a Deligne–Mumford stack. More generally, write $\mathscr{A}_{g,d,n}$ for the analogous moduli functor with $\deg \lambda = d^2$ and $\gcd(d,n) = 1$; its level structure is defined by the same pairing condition. One may also fix a polarization type. The associated λ-Weil pairing on $A[n]$ remains perfect. Over $\mathbb{Z}[1/(dn), \zeta_n]$, this moduli scheme is smooth of relative dimension $g(g+1)/2$.

3. Abelian varieties over finite fields

3.1. **CM abelian varieties and the Honda–Tate theorem.** For an abelian variety A/k, write

$$\mathrm{End}_k^0(A) := \mathrm{End}_k(A) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

An abelian variety A of dimension g is said to *have complex multiplication* if $\mathrm{End}_k^0(A)$ contains a commutative semisimple $\mathbb{Q}$-algebra of dimension $2g$. Specifying its action is additional data. In characteristic zero, a geometrically simple CM abelian variety has an action by a CM field of degree $2g$; a CM field is a totally imaginary quadratic extension of a totally real number field.

Let $q = p^r$. A *Weil q-number* is an algebraic integer π such that

$$|\sigma(\pi)| = q^{1/2}$$

for every embedding $\sigma : \mathbb{Q}(\pi) \hookrightarrow \mathbb{C}$. For $\ell \neq p$, put

$$V_\ell A := \left(\varprojlim_m A[\ell^m](\overline{\mathbb{F}}_q) \right) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell,$$

where the transition maps are multiplication by ℓ. For $A/\mathbb{F}_q$, the q-power Frobenius endomorphism π_A acts on $V_\ell A$ for $\ell \neq p$; its characteristic polynomial $P_A(t) \in \mathbb{Z}[t]$ has degree $2 \dim A$, is independent of ℓ, and its roots are Weil q-numbers.

Theorem 3.1 (Honda–Tate)**.** *The assignment $A \mapsto \pi_A$ gives a bijection between $\mathbb{F}_q$-isogeny classes of simple abelian varieties over $\mathbb{F}_q$ and Galois conjugacy classes of Weil q-numbers.*

The theorem is due to Tate and Honda; see [3, 4]. For a simple A, $\mathrm{End}_{\mathbb{F}_q}^0(A)$ is a division algebra with center $\mathbb{Q}(\pi_A)$, and P_A is a power of the minimal polynomial of π_A. Every abelian variety over $\mathbb{F}_q$ is isogenous to a product of simple ones, so the theorem determines all isogeny classes. The classification is over the specified finite field: extending $\mathbb{F}_q$ to $\mathbb{F}_{q^m}$ replaces π_A by π_A^m and can change the simple factors.

4. Deformation Theory

We study liftings across square-zero thickenings and the cohomology classes that obstruct their existence. Throughout this section, schemes are assumed to be Noetherian and separated. A closed immersion $S \hookrightarrow \widetilde{S}$ defined by an ideal sheaf $\mathcal{I}$ with $\mathcal{I}^2 = 0$ will be called a square-zero thickening. We regard $\mathcal{I}$ as an $\mathcal{O}_S$-module. If $X \to S$ is given, the notation $\mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{I}$ means $\mathcal{F} \otimes_{\mathcal{O}_X} \pi_X^* \mathcal{I}$. The constructions below follow [1, Sec. 2.1].

4.1. **Preliminaries.**

Lemma 4.1. *Let $S \hookrightarrow \widetilde{S}$ be a closed immersion defined by a nilpotent ideal sheaf $\mathcal{I}$, and let $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ be a morphism over $\widetilde{S}$ whose base change $f := \widetilde{f} \times_{\widetilde{S}} S$ is an isomorphism. If $\widetilde{X}$ is flat over $\widetilde{S}$, then $\widetilde{f}$ is an isomorphism.*

Proof. Nilpotent thickenings do not change the underlying topological spaces, so $\widetilde{f}$ is a homeomorphism. A nilpotent thickening of an affine scheme is affine; hence, after restricting to affine opens, it suffices to prove the following module statement. Let $I = \ker(\widetilde{R} \to R)$ be nilpotent, and let $u : N \to M$ be a morphism of $\widetilde{R}$-modules, with M flat over $\widetilde{R}$, such that

$$N/IN \xrightarrow{\sim} M/IM.$$

Put $K := \ker u$ and $Q := \operatorname{coker} u$. Right exactness gives $Q/IQ = 0$, so $Q = IQ = I^2Q = \cdots = 0$. Tensoring the resulting exact sequence $0 \to K \to N \to M \to 0$ with R gives

$$0 = \mathrm{Tor}_1^{\widetilde{R}}(M, R) \longrightarrow K/IK \longrightarrow N/IN \longrightarrow M/IM \longrightarrow 0.$$

Thus $K/IK = 0$, and nilpotence again gives $K = 0$. $\square$

Lemma 4.2. *Let $S \hookrightarrow \widetilde{S}$ be a square-zero thickening. Let $\widetilde{X}$ and $\widetilde{Y}$ be schemes over $\widetilde{S}$, with reductions $X := \widetilde{X} \times_{\widetilde{S}} S$ and $Y := \widetilde{Y} \times_{\widetilde{S}} S$. Assume that $\widetilde{X}$ is flat and $\widetilde{Y}$ is smooth over $\widetilde{S}$. Write*

$$\underline{\mathrm{Der}}_{Y/S} := \underline{\mathrm{Hom}}_{\mathcal{O}_Y}(\Omega^1_{Y/S}, \mathcal{O}_Y).$$

For an S-morphism $f : X \to Y$, let $\mathrm{Mor}_{\widetilde{S}}(\widetilde{X}, \widetilde{Y}, f)$ be the set of $\widetilde{S}$-morphisms reducing to f. This set is either empty or a torsor under

$$H^0\big(X, f^*\underline{\mathrm{Der}}_{Y/S} \otimes_{\mathcal{O}_S} \mathcal{I}\big).$$

Remark 4.3. A torsor under a group G is a nonempty set with a free and transitive G-action. Thus any two liftings differ by a unique element of the displayed group, although there is no preferred lifting to serve as the origin.

Proof. The underlying continuous map of a lifting is already determined by f, so the assertion can be checked locally and then glued. In the affine case, write

$$\widetilde{S} = \operatorname{Spec} \widetilde{R}, \quad S = \operatorname{Spec} R, \quad \widetilde{X} = \operatorname{Spec} \widetilde{M}, \quad \widetilde{Y} = \operatorname{Spec} \widetilde{N},$$

with $I = \ker(\widetilde{R} \to R)$, $M = \widetilde{M}/I\widetilde{M}$, and $N = \widetilde{N}/I\widetilde{N}$. The morphism f corresponds to $u : N \to M$. Fix a lifting $\widetilde{u} : \widetilde{N} \to \widetilde{M}$.

For another lifting $\widetilde{u}'$, the difference $D := \widetilde{u}' - \widetilde{u}$ has image in $I\widetilde{M}$. Flatness of $\widetilde{M}$ gives

$$I\widetilde{M} \simeq \widetilde{M} \otimes_{\widetilde{R}} I \simeq M \otimes_R I.$$

Since both liftings are $\widetilde{R}$-linear and $I^2 = 0$, the map D vanishes on $I\widetilde{N}$ and descends to an R-linear map $N \to M \otimes_R I$. Expanding products and using $(I\widetilde{M})^2 = 0$, we obtain

$$D(n_1 n_2) = u(n_1)D(n_2) + u(n_2)D(n_1).$$

Conversely, every such derivation makes $\widetilde{u} + D$ an algebra homomorphism lifting u. Hence the group of differences is

$$\begin{aligned} \mathrm{Der}_R(N, M \otimes_R I) &\simeq \mathrm{Hom}_N(\Omega^1_{N/R}, M \otimes_R I) \\ &\simeq \mathrm{Hom}_M(M \otimes_{N,u} \Omega^1_{N/R}, M \otimes_R I). \end{aligned}$$

Globally, the corresponding sheaf is

$$\underline{\mathrm{Hom}}_{\mathcal{O}_X}(f^*\Omega^1_{Y/S}, \mathcal{O}_X \otimes_{\mathcal{O}_S} \mathcal{I}) \simeq f^*\underline{\mathrm{Der}}_{Y/S} \otimes_{\mathcal{O}_S} \mathcal{I},$$

where the last isomorphism uses that Y/S is smooth, so $\Omega^1_{Y/S}$ is finite locally free. Taking global sections proves the assertion. □

Corollary 4.4. *Let $\widetilde{Z} \to \widetilde{S}$ be smooth, with reduction Z. The group $\mathrm{Aut}_{\widetilde{S}}(\widetilde{Z}, S)$ of automorphisms inducing Id_Z is canonically isomorphic to the additive group*

$$H^0\big(Z, \underline{\mathrm{Der}}_{Z/S} \otimes_{\mathcal{O}_S} \mathcal{I}\big).$$

Indeed, the identity provides a preferred lifting, and every endomorphism reducing to the identity is an automorphism by the first lemma. Composition corresponds to addition because the product of two first-order corrections is zero.

Corollary 4.5. *Suppose that both $\widetilde{X}$ and $\widetilde{Y}$ are smooth over $\widetilde{S}$. The natural homomorphisms*

$$df : H^0(X, \underline{\mathrm{Der}}_{X/S} \otimes_{\mathcal{O}_S} \mathcal{I}) \longrightarrow H^0(X, f^*\underline{\mathrm{Der}}_{Y/S} \otimes_{\mathcal{O}_S} \mathcal{I})$$

and

$$f^* : H^0(Y, \underline{\mathrm{Der}}_{Y/S} \otimes_{\mathcal{O}_S} \mathcal{I}) \longrightarrow H^0(X, f^*\underline{\mathrm{Der}}_{Y/S} \otimes_{\mathcal{O}_S} \mathcal{I})$$

describe precomposition and postcomposition by automorphisms reducing to the identity. More precisely, if a and b correspond to derivations D_X and D_Y, then

$$\widetilde{f} \circ a = \widetilde{f} + df(D_X), \qquad b \circ \widetilde{f} = \widetilde{f} + f^*(D_Y).$$

Proof. On rings, these assertions follow by composing a lifting $\widetilde{u}$ with maps of the form $1 + D$ and discarding products of first-order corrections. The map df is induced by $f^*\Omega^1_{Y/S} \to \Omega^1_{X/S}$, and f^* is pullback of derivations. □

We will also use the following local lifting facts.

Fact 4.6. A morphism $\widetilde{Y} \to \widetilde{S}$ is smooth at y if and only if, after restricting to a neighborhood of y, it factors as an étale morphism $\widetilde{U} \to \mathbb{A}^r_{\widetilde{S}}$ followed by the projection, for some $r \geq 0$.

Fact 4.7. Base change along a nilpotent closed immersion $S \hookrightarrow \widetilde{S}$ induces an equivalence between the categories of schemes étale over $\widetilde{S}$ and over S.

Fact 4.8. If $X \to S$ is smooth, every point of X has an affine open neighborhood U admitting a smooth lifting $\widetilde{U} \to \widetilde{S}$. If $\widetilde{U}$ and $\widetilde{V}$ lift open subschemes $U, V \subset X$, then on every affine open $W \subset U \cap V$ their restrictions are isomorphic by an isomorphism reducing to Id_W.

For existence, choose local étale coordinates and lift the étale morphism to affine space. For the comparison, smoothness gives a lifting of the identity on an affine source, and the first lemma makes it an isomorphism. The notation $\widetilde{U}|_W$ is meaningful because U and $\widetilde{U}$ have the same underlying topological space.

4.2. **Deformation of smooth schemes.**

Definition 4.9. Let $S \hookrightarrow \widetilde{S}$ be a square-zero thickening and let X/S be smooth. We denote by $\mathrm{Lift}(X; S \hookrightarrow \widetilde{S})$ the set of isomorphism classes of pairs $(\widetilde{X}, \varphi)$, where $\widetilde{X}/\widetilde{S}$ is smooth and $\varphi : \widetilde{X} \times_{\widetilde{S}} S \xrightarrow{\sim} X$. Isomorphisms of liftings are required to respect φ.

Proposition 4.10. *In the setting above, the following hold.*

(1) : *There is a canonically defined obstruction class*

$$o(X; S \hookrightarrow \widetilde{S}) \in H^2(X, \underline{\mathrm{Der}}_{X/S} \otimes_{\mathcal{O}_S} \mathcal{I}),$$

which vanishes if and only if $\mathrm{Lift}(X; S \hookrightarrow \widetilde{S}) \neq \varnothing$.

(2) : *If the obstruction vanishes, this set is a torsor under*

$$H^1(X, \underline{\mathrm{Der}}_{X/S} \otimes_{\mathcal{O}_S} \mathcal{I}).$$

(3) : *An S-isomorphism $f : X \xrightarrow{\sim} Y$ induces, for every i, isomorphisms*

$$df : H^i(X, \underline{\mathrm{Der}}_{X/S} \otimes_{\mathcal{O}_S} \mathcal{I}) \xrightarrow{\sim} H^i(X, f^*\underline{\mathrm{Der}}_{Y/S} \otimes_{\mathcal{O}_S} \mathcal{I}),$$
$$f^* : H^i(Y, \underline{\mathrm{Der}}_{Y/S} \otimes_{\mathcal{O}_S} \mathcal{I}) \xrightarrow{\sim} H^i(X, f^*\underline{\mathrm{Der}}_{Y/S} \otimes_{\mathcal{O}_S} \mathcal{I}),$$

under which $df(o(X; S \hookrightarrow \widetilde{S})) = f^*(o(Y; S \hookrightarrow \widetilde{S}))$.

Proof. Choose an affine open covering $\{U_\alpha\}$ of X and smooth affine liftings $\widetilde{U}_\alpha$ with fixed identifications of their reductions with U_α. Since X is separated, all finite intersections are affine. On $U_{\alpha\beta} := U_\alpha \cap U_\beta$, choose isomorphisms

$$\xi_{\alpha\beta} : \widetilde{U}_\alpha|_{U_{\alpha\beta}} \xrightarrow{\sim} \widetilde{U}_\beta|_{U_{\alpha\beta}}$$

reducing to the identity, with $\xi_{\beta\alpha} = \xi_{\alpha\beta}^{-1}$. On triple intersections put

$$C_{\alpha\beta\gamma} := \xi_{\alpha\gamma}^{-1} \circ \xi_{\beta\gamma} \circ \xi_{\alpha\beta}.$$

This automorphism reduces to the identity, so it defines a section of $\mathcal{T} := \underline{\mathrm{Der}}_{X/S} \otimes_{\mathcal{O}_S} \mathcal{I}$ on $U_{\alpha\beta\gamma}$.

Conjugation by $\xi_{\alpha\beta}$ identifies the automorphism groups on the two charts. This identification is independent of the choice of $\xi_{\alpha\beta}$, since automorphisms reducing to the identity form an abelian group. We may therefore write the sections $C_{\alpha\beta\gamma}$ additively. Associativity of composition on quadruple intersections gives

$$C_{\beta\gamma\delta} - C_{\alpha\gamma\delta} + C_{\alpha\beta\delta} - C_{\alpha\beta\gamma} = 0.$$

Thus C is a Čech 2-cocycle for $\mathcal{T}$.

Replace the transition maps by $\xi'_{\alpha\beta} = \xi_{\alpha\beta} \circ \eta_{\alpha\beta}$, where $\eta_{\alpha\beta}$ reduces to the identity. Under the preceding identifications, direct composition gives

$$C'_{\alpha\beta\gamma} = C_{\alpha\beta\gamma} + \eta_{\beta\gamma} - \eta_{\alpha\gamma} + \eta_{\alpha\beta} = C_{\alpha\beta\gamma} + (\partial\eta)_{\alpha\beta\gamma}.$$

Hence $[C]$ is independent of the transition maps. Different local liftings are locally isomorphic, and transporting their transition maps reduces to the same calculation. Refining the cover also preserves the class. Since affine intersections are acyclic for the quasi-coherent sheaf $\mathcal{T}$, this gives a well-defined class

$$o(X; S \hookrightarrow \widetilde{S}) := [C] \in H^2(X, \mathcal{T}).$$

If this class is zero, modify the transition maps by a 1-cochain whose coboundary is $-C$. The resulting maps satisfy $\xi'_{\beta\gamma} \circ \xi'_{\alpha\beta} = \xi'_{\alpha\gamma}$ and glue the local liftings to a smooth scheme $\widetilde{X}/\widetilde{S}$ lifting X. Conversely, the transition maps obtained from a global lifting satisfy the cocycle condition, so its obstruction is zero. This proves (1).

For (2), fix one lifting $\widetilde{X}$ and its transition maps $\xi_{\alpha\beta}$. If $d = \{d_{\alpha\beta}\}$ is a 1-cocycle for $\mathcal{T}$, then

$$\xi^d_{\alpha\beta} := \xi_{\alpha\beta} \circ d_{\alpha\beta}$$

again satisfies the cocycle condition and defines a lifting $\widetilde{X}^{,d}$. Here $d_{\alpha\beta}$ is viewed as the corresponding automorphism. Replacing d by $d + \partial e$ changes the gluing by chart automorphisms, so gives an isomorphic marked lifting. Conversely, an isomorphism between two such liftings restricts to chart automorphisms and shows that their cocycles differ by a coboundary. Thus the action of $H^1(X, \mathcal{T})$ on liftings is free.

To see transitivity, let $\widetilde{X}'$ be another lifting. After refining the cover, choose isomorphisms $\widetilde{U}_\alpha \xrightarrow{\sim} \widetilde{X}'|_{U_\alpha}$ reducing to the identity. Transporting the transition maps of $\widetilde{X}'$ to the charts of $\widetilde{X}$ gives maps differing from $\xi_{\alpha\beta}$ by a 1-cocycle d. Hence $\widetilde{X}' \simeq \widetilde{X}^{,d}$, proving transitivity.

For (3), use the cover $\{f^{-1}(U_\alpha)\}$ of X obtained from an affine cover of Y. Regard a lifting of each U_α as a lifting of $f^{-1}(U_\alpha)$ by transporting its identification of the special fiber through f. Conjugation of the transition maps carries the obstruction cocycle on Y to the one on X. On derivations this is precisely $(df)^{-1} \circ f^*$, which proves the claimed identity. □

4.3. **Deformation of morphisms.**

Definition 4.11. Let $S \hookrightarrow \widetilde{S}$ be a square-zero thickening and let $f : X \to Y$ be a morphism between smooth S-schemes. Fix smooth liftings $\widetilde{X}$ and $\widetilde{Y}$, together with their identifications over S. We denote by $\mathrm{Lift}(f; \widetilde{X}, \widetilde{Y}; S \hookrightarrow \widetilde{S})$ the set of $\widetilde{S}$-morphisms $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ reducing to f.

Proposition 4.12. (1) : *There is a canonically defined obstruction*

$$o(f; \widetilde{X}, \widetilde{Y}; S \hookrightarrow \widetilde{S}) \in H^1(X, f^*\underline{\mathrm{Der}}_{Y/S} \otimes_{\mathcal{O}_S} \mathcal{I}),$$

which vanishes if and only if $\mathrm{Lift}(f; \widetilde{X}, \widetilde{Y}; S \hookrightarrow \widetilde{S})$ *is nonempty.*

(2) : *When it is nonempty, this set is a torsor under*

$$H^0(X, f^*\underline{\mathrm{Der}}_{Y/S} \otimes_{\mathcal{O}_S} \mathcal{I}).$$

(3) : *Write* $\widetilde{X}' = \widetilde{X} + m_{\widetilde{X}}$ *and* $\widetilde{Y}' = \widetilde{Y} + m_{\widetilde{Y}}$ *for the actions of the preceding torsors. Under the natural maps*

$$df : H^1(X, \underline{\mathrm{Der}}_{X/S} \otimes_{\mathcal{O}_S} \mathcal{I}) \longrightarrow H^1(X, f^*\underline{\mathrm{Der}}_{Y/S} \otimes_{\mathcal{O}_S} \mathcal{I}),$$
$$f^* : H^1(Y, \underline{\mathrm{Der}}_{Y/S} \otimes_{\mathcal{O}_S} \mathcal{I}) \longrightarrow H^1(X, f^*\underline{\mathrm{Der}}_{Y/S} \otimes_{\mathcal{O}_S} \mathcal{I}),$$

the obstructions satisfy

$$\begin{aligned} &o(f; \widetilde{X} + m_{\widetilde{X}}, \widetilde{Y} + m_{\widetilde{Y}}; S \hookrightarrow \widetilde{S}) \\ &\quad = o(f; \widetilde{X}, \widetilde{Y}; S \hookrightarrow \widetilde{S}) + df(m_{\widetilde{X}}) - f^*(m_{\widetilde{Y}}). \end{aligned}$$

Proof. Choose affine covers $\{U_\alpha\}$ of X and $\{V_\alpha\}$ of Y, refining the indexing if necessary, such that $f(U_\alpha) \subset V_\alpha$. Smoothness of $\widetilde{Y}$ gives local liftings

$$\widetilde{f}_\alpha : \widetilde{U}_\alpha := \widetilde{X}|_{U_\alpha} \longrightarrow \widetilde{V}_\alpha := \widetilde{Y}|_{V_\alpha}.$$

Let $\xi^X_{\alpha\beta}$ and $\xi^Y_{\alpha\beta}$ be the transition maps for these charts. On $U_{\alpha\beta}$, compare $\widetilde{f}_\alpha$ with the transported lifting

$$(\xi^Y_{\alpha\beta})^{-1} \circ \widetilde{f}_\beta \circ \xi^X_{\alpha\beta}.$$

Their difference, in the torsor of local liftings, is a section

$$c_{\alpha\beta} := (\xi^Y_{\alpha\beta})^{-1} \circ \widetilde{f}_\beta \circ \xi^X_{\alpha\beta} - \widetilde{f}_\alpha \in H^0(U_{\alpha\beta}, \mathcal{E}), \qquad \mathcal{E} := f^*\underline{\mathrm{Der}}_{Y/S} \otimes_{\mathcal{O}_S} \mathcal{I}.$$

Because both chart systems satisfy the cocycle condition, $c_{\alpha\gamma} = c_{\alpha\beta} + c_{\beta\gamma}$. Replacing $\widetilde{f}_\alpha$ by $\widetilde{f}_\alpha + e_\alpha$ changes c to $c + \partial e$, where $(\partial e)_{\alpha\beta} = e_\beta - e_\alpha$. Thus $[c] \in H^1(X, \mathcal{E})$ is independent of the local liftings and, by refinement, of the covers.

If $[c] = 0$, choose e with $\partial e = -c$. The maps $\widetilde{f}_\alpha + e_\alpha$ then agree on overlaps and glue to a global lifting. Conversely, a global lifting makes the obstruction cocycle zero. This proves (1) with $o(f; \widetilde{X}, \widetilde{Y}; S \hookrightarrow \widetilde{S}) := [c]$. Any two global liftings differ by a unique global section of $\mathcal{E}$, proving (2).

For (3), represent $m_{\widetilde{X}}$ and $m_{\widetilde{Y}}$ by 1-cocycles and replace the transition maps by

$$(\xi^X_{\alpha\beta})' = \xi^X_{\alpha\beta} \circ \eta^X_{\alpha\beta}, \qquad (\xi^Y_{\alpha\beta})' = \xi^Y_{\alpha\beta} \circ \eta^Y_{\alpha\beta},$$

where the η's are the corresponding infinitesimal automorphisms. The transported map becomes

$$(\eta^Y_{\alpha\beta})^{-1} \circ (\xi^Y_{\alpha\beta})^{-1} \circ \widetilde{f}_\beta \circ \xi^X_{\alpha\beta} \circ \eta^X_{\alpha\beta}.$$

Precomposition adds $df(m_{\widetilde{X},\alpha\beta})$, and postcomposition by the inverse adds $-f^*(m_{\widetilde{Y},\alpha\beta})$. Consequently,

$$c'_{\alpha\beta} = c_{\alpha\beta} + df(m_{\widetilde{X},\alpha\beta}) - f^*(m_{\widetilde{Y},\alpha\beta}),$$

giving the stated formula in cohomology. □

Remark 4.13. These constructions are compatible with base change. Suppose we have a commutative diagram

$$\begin{array}{ccc} S & \hookrightarrow & \widetilde{S} \\ \downarrow & & \downarrow \\ T & \hookrightarrow & \widetilde{T}, \end{array}$$

where the horizontal immersions have square-zero ideals $\mathcal{I}$ and $\mathcal{J}$. The vertical morphisms induce $\mathcal{J} \to \mathcal{I}$ after pullback. A smooth lifting over $\widetilde{T}$ pulls back to one over $\widetilde{S}$; the obstruction classes and the actions on liftings pull back via the induced maps on cohomology and on these ideal sheaves. This follows by pulling back the local transition maps and their cocycles. The square need not be Cartesian.

4.4. **Deformation of invertible sheaves.**

Definition 4.14. Let $S \hookrightarrow \widetilde{S}$ be a square-zero thickening, let X/S be smooth, and fix a smooth lifting $\widetilde{X}/\widetilde{S}$. For a line bundle $\mathcal{L}$ on X, let $\mathrm{Lift}(\mathcal{L}; \widetilde{X}; S \hookrightarrow \widetilde{S})$ be the set of isomorphism classes of pairs $(\widetilde{\mathcal{L}}, \psi)$, where $\widetilde{\mathcal{L}}$ is a line bundle on $\widetilde{X}$ and $\psi : \widetilde{\mathcal{L}}|_X \xrightarrow{\sim} \mathcal{L}$. Isomorphisms are required to respect ψ.

Proposition 4.15. (1) : *There is a canonical obstruction class*

$$o(\mathcal{L}; \widetilde{X}; S \hookrightarrow \widetilde{S}) \in H^2(X, \mathcal{O}_X \otimes_{\mathcal{O}_S} \mathcal{I}),$$

which vanishes if and only if $\mathrm{Lift}(\mathcal{L}; \widetilde{X}; S \hookrightarrow \widetilde{S})$ *is nonempty.*

(2) : *When the obstruction vanishes, the set of marked liftings in the definition is a torsor under* $H^1(X, \mathcal{O}_X \otimes_{\mathcal{O}_S} \mathcal{I})$. *If*

$$H^0(\widetilde{X}, \mathcal{O}^\times_{\widetilde{X}}) \longrightarrow H^0(X, \mathcal{O}^\times_X)$$

is surjective, the same assertion holds after forgetting ψ.

(3) : *The homomorphism*

$$\mathrm{dlog} : \mathrm{Pic}(X) = H^1(X, \mathcal{O}^\times_X) \longrightarrow H^1(X, \Omega^1_{X/S})$$

is induced by $a \mapsto a^{-1}da$. *Cup product with* $\mathrm{dlog}(\mathcal{L})$, *followed by evaluation of derivations on differentials, defines*

$$d_{\mathcal{L}} : H^i(X, \underline{\mathrm{Der}}_{X/S} \otimes_{\mathcal{O}_S} \mathcal{I}) \longrightarrow H^{i+1}(X, \underline{\mathrm{Der}}_{X/S} \otimes_{\mathcal{O}_X} \Omega^1_{X/S} \otimes_{\mathcal{O}_S} \mathcal{I}) \longrightarrow H^{i+1}(X, \mathcal{O}_X \otimes_{\mathcal{O}_S} \mathcal{I}).$$

In degree $i = 1$, it satisfies

$$o(\mathcal{L}; \widetilde{X} + m_{\widetilde{X}}; S \hookrightarrow \widetilde{S}) = o(\mathcal{L}; \widetilde{X}; S \hookrightarrow \widetilde{S}) + d_{\mathcal{L}}(m_{\widetilde{X}}).$$

Proof. Put $\mathcal{J} := \mathcal{O}_X \otimes_{\mathcal{O}_S} \mathcal{I}$. Flatness identifies $\mathcal{J}$ with the ideal of X in $\widetilde{X}$. Since $\mathcal{J}^2 = 0$, the map $a \mapsto 1 + a$ is an isomorphism from its additive group to $1 + \mathcal{J}$. On the common underlying topological space, we obtain an exact sequence of sheaves of abelian groups

$$0 \longrightarrow \mathcal{J} \xrightarrow{a \mapsto 1+a} \mathcal{O}_{\widetilde{X}}^{\times} \longrightarrow \mathcal{O}_X^{\times} \longrightarrow 1.$$

Its long exact sequence contains

$$\begin{aligned} H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}^{\times}) \longrightarrow H^0(X, \mathcal{O}_X^{\times}) \longrightarrow H^1(X, \mathcal{J}) \\ \longrightarrow \operatorname{Pic}(\widetilde{X}) \longrightarrow \operatorname{Pic}(X) \xrightarrow{\mathrm{ob}} H^2(X, \mathcal{J}). \end{aligned}$$

Define $o(\mathcal{L}; \widetilde{X}; S \hookrightarrow \widetilde{S})$ to be the image of $[\mathcal{L}]$ under the connecting homomorphism ob. Exactness proves (1).

For a direct description, choose an affine cover trivializing $\mathcal{L}$, with transition functions $l_{\alpha\beta}$ satisfying $l_{\alpha\gamma}^{-1} l_{\beta\gamma} l_{\alpha\beta} = 1$. Choose units $\widetilde{l}_{\alpha\beta}$ lifting them on the corresponding charts of $\widetilde{X}$. The defects

$$h_{\alpha\beta\gamma} := \widetilde{l}_{\alpha\gamma}^{-1} \xi_{\alpha\beta}^*(\widetilde{l}_{\beta\gamma}) \widetilde{l}_{\alpha\beta} - 1 \in \mathcal{J}(U_{\alpha\beta\gamma})$$

form a 2-cocycle representing this obstruction. Changing the lifted units changes h by a coboundary.

Once a marked lifting is fixed, the ratio of the transition functions of any other marked lifting to its transition functions is a 1-cocycle with values in $1 + \mathcal{J} \simeq \mathcal{J}$. Changing trivializations compatibly with the marking changes this cocycle by a coboundary. Thus the marked liftings form a torsor under $H^1(X, \mathcal{J})$. For unmarked liftings, their isomorphism classes form a torsor under the image of $H^1(X, \mathcal{J})$ in $\operatorname{Pic}(\widetilde{X})$; the stated surjectivity makes this map injective. This proves (2).

Finally, represent $m_{\widetilde{X}}$ by derivations $D_{\alpha\beta}$ and replace $\xi_{\alpha\beta}$ by $\xi'_{\alpha\beta} = \xi_{\alpha\beta} \circ \eta_{\alpha\beta}$, with $\eta_{\alpha\beta}^* = 1 + D_{\alpha\beta}$. Keeping the same lifted transition functions, we have

$$(\xi'_{\alpha\beta})^*(\widetilde{l}_{\beta\gamma}) = \xi_{\alpha\beta}^*(\widetilde{l}_{\beta\gamma})\big(1 + D_{\alpha\beta}(\operatorname{dlog} l_{\beta\gamma})\big),$$

where a derivation is viewed as a homomorphism from $\Omega^1_{X/S}$ to $\mathcal{J}$. Products of sections of $\mathcal{J}$ vanish, so

$$h'_{\alpha\beta\gamma} - h_{\alpha\beta\gamma} = D_{\alpha\beta}(\operatorname{dlog} l_{\beta\gamma}).$$

This is the Čech representative of the evaluated cup product $m_{\widetilde{X}} \smile \operatorname{dlog}(\mathcal{L})$. Passing to cohomology proves (3). □

4.5. **De Rham cohomology and its deformation.** Let $\pi : X \to S$ be a morphism of schemes. The relative de Rham complex is

$$\Omega^{\bullet}_{X/S} : \mathcal{O}_X \xrightarrow{d} \Omega^1_{X/S} \xrightarrow{d} \Omega^2_{X/S} \xrightarrow{d} \cdots, \qquad \Omega^q_{X/S} := \bigwedge_{\mathcal{O}_X}^{q} \Omega^1_{X/S}.$$

Its differentials are $\pi^{-1}\mathcal{O}_S$-linear, although they are generally not $\mathcal{O}_X$-linear. The relative algebraic de Rham cohomology sheaves are

$$\underline{H}^i_{\mathrm{dR}}(X/S) := \mathcal{H}^i(R\pi_* \Omega^{\bullet}_{X/S}).$$

When $S = \operatorname{Spec} k$, we write $H^i_{\mathrm{dR}}(X/k) := \mathbb{H}^i(X, \Omega^{\bullet}_{X/k})$.

For the following Čech computations, suppose that π is smooth, separated, and quasi-compact. Locally on S, we may assume that S is affine and choose an affine open covering $\{U_\alpha\}_\alpha$ such that each U_α admits an étale morphism to some $\mathbb{A}^r_S$. If $x_1, \ldots, x_r$ are the pulled-back coordinates, then

$$\Omega^1_{U_\alpha/S} = \bigoplus_{j=1}^{r} \mathcal{O}_{U_\alpha}\, dx_j.$$

Finite intersections of these affine opens are affine because X/S is separated. Hence their higher cohomology with coefficients in $\Omega^q_{X/S}$ vanishes.

We can therefore compute de Rham cohomology from the double complex

$$C^{p,q} := C^p(\{U_\alpha\}_\alpha, \Omega^q_{X/S}) = \prod_{\alpha_0<\cdots<\alpha_p} \Gamma(U_{\alpha_0\cdots\alpha_p}, \Omega^q_{X/S}),$$

with differentials

$$\partial : C^{p,q} \longrightarrow C^{p+1,q}, \qquad (\partial x)_{\alpha_0\cdots\alpha_{p+1}} = \sum_{j=0}^{p+1} (-1)^j x_{\alpha_0\cdots\widehat{\alpha_j}\cdots\alpha_{p+1}}\big|_{U_{\alpha_0\cdots\alpha_{p+1}}},$$

and

$$d : C^{p,q} \longrightarrow C^{p,q+1}, \qquad (x_{\alpha_0\cdots\alpha_p}) \longmapsto (dx_{\alpha_0\cdots\alpha_p}).$$

Let $\mathscr{C}^\bullet$ be its total complex:

$$\mathscr{C}^n = \bigoplus_{p+q=n} C^{p,q}, \qquad D|_{C^{p,q}} = \partial + (-1)^p d.$$

The same construction on inverse images of affine open subsets of S gives the relative complex of sheaves.

Fact 4.16. The differentials satisfy $\partial d = d\partial$. The sign in the definition of D consequently gives $D^2 = 0$.

Proposition 4.17. *The augmentation from $\Omega^\bullet_{X/S}$ to its Čech resolution is a quasi-isomorphism. In particular, on every affine open subset of S, the complex $\mathscr{C}^\bullet$ computes the relative de Rham cohomology:*

$$\underline{H}^i_{\mathrm{dR}}(X/S) \simeq \mathcal{H}^i(\mathscr{C}^\bullet).$$

The augmentation is a quasi-isomorphism in each fixed differential degree, and the affine intersections are acyclic for the corresponding quasi-coherent sheaves. Taking the total complex proves the assertion; see [1, Sec. 2.1.6].

Now let $S \hookrightarrow \widetilde{S}$ be a closed immersion defined by $\mathcal{I}$ with $\mathcal{I}^2 = 0$. Let $\widetilde{X}$ be smooth, separated, and quasi-compact over $\widetilde{S}$, and put $X := \widetilde{X} \times_{\widetilde{S}} S$. We may choose an affine open covering $\{\widetilde{U}_\alpha\}_\alpha$ whose reductions U_α have the preceding properties. The terms of the Čech–de Rham complex are determined by these local liftings. Its Čech differential also depends on their gluing isomorphisms

$$\xi_{\alpha\beta} : \widetilde{U}_\alpha|_{U_{\alpha\beta}} \xrightarrow{\sim} \widetilde{U}_\beta|_{U_{\alpha\beta}}.$$

Proposition 4.18. *Let*

$$m_{\widetilde{X}} \in H^1\big(X, \underline{\mathrm{Der}}_{X/S} \otimes_{\mathcal{O}_S} \mathcal{I}\big), \qquad \widetilde{X}' = \widetilde{X} + m_{\widetilde{X}} \in \mathrm{Lift}(X; S \hookrightarrow \widetilde{S}).$$

There is a canonical isomorphism

$$\underline{H}^1_{\mathrm{dR}}(\widetilde{X}/\widetilde{S}) \xrightarrow{\sim} \underline{H}^1_{\mathrm{dR}}(\widetilde{X}'/\widetilde{S})$$

compatible with the identity on $\underline{H}^1_{\mathrm{dR}}(X/S)$ under restriction to X.

Proof. The question is local on $\widetilde{S}$, so we work over an affine base. Use the same local liftings $\widetilde{U}_\alpha$ for both schemes. Their gluing isomorphisms differ by

$$\xi'_{\alpha\beta} = \xi_{\alpha\beta} \circ \eta_{\alpha\beta}, \qquad \eta^*_{\alpha\beta}|_{\mathcal{O}} = \mathrm{Id} + T_{\alpha\beta} \circ d,$$

where

$$T_{\alpha\beta} \in \mathrm{Hom}_{\mathcal{O}_{U_{\alpha\beta}}}\big(\Omega^1_{U_{\alpha\beta}/S}, \mathcal{O}_{U_{\alpha\beta}} \otimes_{\mathcal{O}_S} \mathcal{I}\big)$$

form a Čech cocycle representing $m_{\widetilde{X}}$. In all formulas below, restrictions and transport by the original gluing maps are understood.

Write ι_T for contraction by such an $\mathcal{I}$-valued vector field. On differential forms the infinitesimal automorphism acts by

$$\eta^* = \mathrm{Id} + \mathcal{L}_T, \qquad \mathcal{L}_T = d\iota_T + \iota_T d.$$

Indeed, on functions it acts by $a \mapsto a + T(da)$, and on their differentials it acts by $da \mapsto da + d(T(da))$. These identities determine the action on all forms, since products of two $\mathcal{I}$-valued terms vanish.

Under the local identifications of the two double complexes, the vertical differential d is unchanged, while

$$\partial' = \partial + \mathcal{L}_T \smile (-).$$

Here the contraction cup product has bidegree $(1,-1)$:

$$(\iota_T \smile x)_{\alpha_0\cdots\alpha_{p+1}} = \iota_{T_{\alpha_0\alpha_1}} x_{\alpha_1\cdots\alpha_{p+1}}.$$

Because $\partial T = 0$, it satisfies

$$\partial(\iota_T \smile x) = -\iota_T \smile \partial x.$$

A total 1-cochain is a pair $(f,\omega) \in C^{1,0} \oplus C^{0,1}$. It is closed precisely when

$$\partial f = 0, \qquad \partial\omega = df, \qquad d\omega = 0.$$

Define

$$(f,\omega) \longmapsto (f',\omega') := \big(f + \iota_T \smile \omega, \omega\big).$$

For a closed pair, the preceding identities give

$$\begin{aligned} \partial' f' &= \partial f + \iota_T \smile df - \iota_T \smile \partial\omega = 0, \\ \partial'\omega' - df' &= \partial\omega - df + \mathcal{L}_T \smile \omega - d(\iota_T \smile \omega) = \iota_T \smile d\omega = 0, \\ d\omega' &= 0. \end{aligned}$$

Moreover, a boundary $(\partial a, da)$ is sent to

$$(\partial a + \iota_T \smile da, da) = (\partial' a, da),$$

which is a boundary for the second complex. Thus the formula induces a map on H^1. Replacing T by $-T$ gives its inverse, since all products of two $\mathcal{I}$-valued corrections vanish.

Changing the local identifications changes T by a coboundary ∂v. Cartan's formula shows that, after making the corresponding local identifications, the resulting maps on closed pairs differ by a total boundary given by contraction with v. The construction is consequently independent of these choices and of refinements of the covering. Its correction term is $\mathcal{I}$-valued, so it restricts to the identity on X. The local isomorphisms therefore glue over $\widetilde{S}$. □

4.6. **Infinitesimal deformation.** Let Λ be a complete Noetherian local ring with residue field k. Following [1, App. B.1], let $\mathcal{C}_\Lambda$ denote the category of Artinian local Λ-algebras with residue field identified with k, and let $\widehat{\mathcal{C}}_\Lambda$ denote the category of complete Noetherian local Λ-algebras with the same residue field. Morphisms are continuous local Λ-algebra homomorphisms inducing the identity on k.

For $(R,\mathfrak{m}) \in \widehat{\mathcal{C}}_\Lambda$, we have

$$R \simeq \varprojlim_{r\geq 0} R/\mathfrak{m}^{r+1}, \qquad R/\mathfrak{m}^{r+1} \in \mathcal{C}_\Lambda.$$

A covariant functor $F : \mathcal{C}_\Lambda \to \mathsf{Set}$ extends to $\widehat{F} : \widehat{\mathcal{C}}_\Lambda \to \mathsf{Set}$ by

$$\widehat{F}(R) := \varprojlim_{r\geq 0} F(R/\mathfrak{m}^{r+1}).$$

If F is already defined on $\widehat{\mathcal{C}}_\Lambda$, restriction to the Artinian quotients gives a canonical map

$$F(R) \longrightarrow \widehat{F}(R) := \varprojlim_{r\geq 0} F(R/\mathfrak{m}^{r+1}).$$

This map need not be bijective. For $R \in \widehat{\mathcal{C}}_\Lambda$, define

$$h_R : \mathcal{C}_\Lambda \longrightarrow \mathsf{Set}, \qquad h_R(A) := \mathrm{Hom}_{\Lambda,\mathrm{cont}}(R,A).$$

Lemma 4.19. *For every $F : \mathcal{C}_\Lambda \to \mathsf{Set}$ and $R \in \widehat{\mathcal{C}}_\Lambda$, there is a canonical bijection*

$$\widehat{F}(R) \simeq \mathrm{Nat}(h_R, F).$$

A compatible system $(\xi_r)_r$ defines a natural transformation by sending $u : R \to A$ to $F(u_r)(\xi_r)$, where $u_r : R/\mathfrak{m}^{r+1} \to A$ is a factorization of u. Such a factorization exists because A is Artinian, and compatibility makes the result independent of r. Conversely, evaluate a natural transformation on the quotient maps $R \to R/\mathfrak{m}^{r+1}$.

Definition 4.20. A functor $F : \mathcal{C}_\Lambda \to \mathsf{Set}$ is *prorepresentable* if there exist $R \in \widehat{\mathcal{C}}_\Lambda$ and $\widehat{\xi} \in \widehat{F}(R)$ inducing an isomorphism

$$\widehat{\xi} : h_R \xrightarrow{\sim} F.$$

The pair $(R, \widehat{\xi})$ is a universal formal object for F.

Definition 4.21. Let $F : \widehat{\mathcal{C}}_\Lambda \to \mathsf{Set}$ be a functor whose restriction to $\mathcal{C}_\Lambda$ is prorepresented by $(R, \widehat{\xi})$. It is *effectively prorepresentable* if there exists $\xi \in F(R)$ mapping to $\widehat{\xi} \in \widehat{F}(R)$.

Let $F : \widehat{\mathcal{C}}_\Lambda \to \mathsf{Set}$ and fix $\xi_0 \in F(k)$.

Definition 4.22. An *infinitesimal deformation* of ξ_0 over $A \in \mathcal{C}_\Lambda$ is an element $\eta \in F(A)$ whose image under $F(A) \to F(k)$ is ξ_0.

Definition 4.23. A *formal deformation* of ξ_0 over $(R, \mathfrak{m}) \in \widehat{\mathcal{C}}_\Lambda$ is a compatible system

$$\widehat{\xi} = (\xi_r)_{r \geq 0} \in \widehat{F}(R) = \varprojlim_{r \geq 0} F(R/\mathfrak{m}^{r+1})$$

whose image in $F(k)$ is the prescribed ξ_0.

Definition 4.24. A formal deformation $\widehat{\xi} \in \widehat{F}(R)$ is *effective* if it is induced by an element $\xi \in F(R)$.

Recall the moduli problem $\mathscr{A}_{g,d,n}$ for polarized abelian schemes with level structure. Its objects over a scheme T are tuples (A, λ, α_n) of the prescribed type, and $\mathscr{A}_{g,d,n}(T)$ denotes their isomorphism classes. If this functor is represented by a scheme, then

$$\mathscr{A}_{g,d,n}(T) \simeq \mathrm{Mor}(T, \mathscr{A}_{g,d,n}).$$

Thus such objects are the T-points of the moduli scheme. Even before representability is known, Yoneda identifies an element of $\mathscr{A}_{g,d,n}(T)$ with a natural transformation

$$h_T \longrightarrow \mathscr{A}_{g,d,n}.$$

Let k have characteristic p, with $p \nmid dn$ when $p > 0$. Set $\Lambda = k$ if $p = 0$. If $p > 0$, assume k is perfect and set $\Lambda = W(k)$. Write $\mathcal{C} = \mathcal{C}_\Lambda$ and $\widehat{\mathcal{C}} = \widehat{\mathcal{C}}_\Lambda$. Fix an object

$$\xi_0 = (A_0, \lambda_0, \alpha_{n,0}) \quad \text{over } k.$$

We define the marked deformation functor

$$\mathrm{Def}_{\xi_0} : \widehat{\mathcal{C}} \longrightarrow \mathsf{Set}, \qquad R \longmapsto \{(\xi, f_0)\}/\cong,$$

where $\xi = (A, \lambda, \alpha_n)$ is an object over R and

$$f_0 : \xi \otimes_R k \xrightarrow{\sim} \xi_0$$

is an isomorphism preserving all the structures. An isomorphism $(\xi, f_0) \to (\xi', f_0')$ is an isomorphism $h : \xi \to \xi'$ satisfying $f_0' \circ (h \otimes_R k) = f_0$.

We also consider deformation functors obtained by retaining fewer structures:

(1) : $\mathrm{Def}_{A_0}(R) = \{(A, f_0)\}/\cong$, where A/R is an abelian scheme and $f_0 : A \otimes_R k \xrightarrow{\sim} A_0$ is an isomorphism of abelian varieties.

(2) : $\mathrm{Def}_{(A_0,\lambda_0)}(R) = \{(A, \lambda, f_0)\}/\cong$, where $\lambda : A \to A^\vee$ is a polarization and

$$f_0^\vee \circ \lambda_0 \circ f_0 = \lambda \otimes_R k.$$

(3) : $\mathrm{Def}_{\xi_0} = \mathrm{Def}_{(A_0,\lambda_0,\alpha_{n,0})}$ also requires f_0 to preserve the level structure.

4.7. **Schlessinger's criterion for prorepresentability.**

Definition 4.25. A surjection $p : \widetilde{R} \to R$ in $\mathcal{C}$ is a *small surjection* if its kernel I satisfies

$$I\mathfrak{m}_{\widetilde{R}} = 0.$$

A covariant functor $F : \mathcal{C} \to \mathsf{Set}$ is representable if it is isomorphic to $\mathrm{Hom}_{\mathcal{C}}(R, -)$ for some $R \in \mathcal{C}$; it is prorepresentable if it is isomorphic to h_R for some $R \in \widehat{\mathcal{C}}$.

Remark 4.26. Every surjection in $\mathcal{C}$ factors as a finite composition of small surjections. For a small surjection, I is a finite-dimensional k-vector space and $I^2 = 0$.

If $F : \mathcal{C} \to \mathsf{Set}$ is prorepresentable, then:
(1) : $F(k)$ has exactly one element, since every object has its specified residue map to k.
(2) : For every surjection $\widetilde{R} \to R$ and every morphism $Q \to R$ in $\mathcal{C}$, the natural map is a bijection

$$F(Q \times_R \widetilde{R}) \xrightarrow{\sim} F(Q) \times_{F(R)} F(\widetilde{R}).$$

Theorem 4.27. *Let $F : \mathcal{C} \to \mathsf{Set}$ be covariant with $F(k)$ a singleton. It is prorepresentable if and only if it satisfies* (2) *and its tangent space*

$$t_F := F(k[\epsilon]/(\epsilon^2))$$

is finite-dimensional over k. Here the k-vector space structure on t_F is the one induced by the fiber-product property. It suffices to check (2) *when $\widetilde{R} \to R$ is small.*

Moreover, if F is prorepresented by R^{univ} and is formally smooth, meaning that $F(\widetilde{R}) \to F(R)$ is surjective for every surjection in $\mathcal{C}$, then, writing $m = \dim_k t_F$, there is a (generally noncanonical) isomorphism

$$R^{\mathrm{univ}} \simeq \Lambda[[t_1, \ldots, t_m]].$$

This is Schlessinger's criterion in the form used here; see [1, Thm. 2.2.1.4].

Remark 4.28. A ring map $A \to B$ is *formally smooth* if, for every A-algebra C and every nilpotent ideal $J \subset C$, every A-algebra map $B \to C/J$ lifts to $B \to C$. For complete local rings in the preceding criterion, we use the corresponding lifting property for continuous local homomorphisms to Artinian local rings.

Remark 4.29. We write $o(X; \widetilde{R} \to R)$ instead of $o(X; S \hookrightarrow \widetilde{S})$ when $S = \operatorname{Spec} R$ and $\widetilde{S} = \operatorname{Spec} \widetilde{R}$. For a small surjection with kernel I, put $S_0 = \operatorname{Spec} k$ and $X_0 = X \otimes_R k$. Since $\mathfrak{m}_{\widetilde{R}} I = 0$, the obstruction and deformation groups can be written as

$$H^i(X_0, \underline{\mathrm{Der}}_{X_0/k}) \otimes_k I$$

in place of $H^i(X, \underline{\mathrm{Der}}_{X/S} \otimes_R I)$.

Remark 4.30. The deformation theory of smooth schemes initially forgets the group law on an abelian scheme. The following results explain how rigidity recovers that structure; compare [1, Sec. 2.2.2].

Lemma 4.31. *Let $S \hookrightarrow \widetilde{S}$ be a closed immersion defined by a nilpotent ideal sheaf. Let $\widetilde{A}$ and $\widetilde{A}'$ be abelian schemes over $\widetilde{S}$, with reductions A and A' over S. Restriction induces an injection*

$$\mathrm{Hom}_{\widetilde{S}}(\widetilde{A}, \widetilde{A}') \hookrightarrow \mathrm{Hom}_S(A, A'),$$

where Hom *means group scheme homomorphisms. The same holds for* Isom.

Proof. If f and g have the same reduction, their difference $f - g$ is trivial on A. By rigidity, it factors through a section $\eta : \widetilde{S} \to \widetilde{A}'$. Since f and g preserve the identity sections, η is the identity section, so $f = g$. □

Remark 4.32. In particular, an abelian scheme has no nontrivial infinitesimal automorphisms as a group scheme. Its underlying scheme can still have infinitesimal translations.

Corollary 4.33. *With the preceding notation, suppose a group scheme homomorphism $f : A \to A'$ lifts to a morphism of schemes $\widetilde{f} : \widetilde{A} \to \widetilde{A}'$ over $\widetilde{S}$. Then*

$$\widetilde{f} - \widetilde{f} \circ e_{\widetilde{A}}$$

is the unique group scheme homomorphism lifting f. In this formula the section $\widetilde{f} \circ e_{\widetilde{A}}$ is pulled back along the structure map of $\widetilde{A}$.

Indeed, the translated morphism preserves the identity and is therefore a group homomorphism by rigidity. Its translation term restricts to zero over S, and uniqueness follows from the lemma.

Proposition 4.34. *Let $p : \widetilde{R} \to R$ be a small surjection in $\mathcal{C}$ and let (A, f_0) represent an element of $\mathrm{Def}_{A_0}(R)$. Forgetting the group structure induces a bijection*

$$\mathrm{Def}_{A_0}(p)^{-1}([(A, f_0)]) \xrightarrow{\sim} \mathrm{Lift}(A; \widetilde{R} \to R),$$

where the right-hand side consists of marked liftings of A as a smooth scheme.

Proof. For $(\widetilde{A}, \widetilde{f}_0)$ in the left-hand side, choose a group scheme isomorphism

$$\psi : \widetilde{A} \otimes_{\widetilde{R}} R \xrightarrow{\sim} A, \qquad \psi \otimes_R k = f_0^{-1} \circ \widetilde{f}_0.$$

The rigidity lemma above makes ψ unique. If an isomorphism $h : (\widetilde{A}, \widetilde{f}_0) \to (\widetilde{A}', \widetilde{f}_0')$ is given, rigidity also gives $\psi = \psi' \circ (h \otimes_{\widetilde{R}} R)$. Thus the construction is well defined on isomorphism classes.

For injectivity, suppose $h : (\widetilde{A}, \psi) \to (\widetilde{A}', \psi')$ is an isomorphism of the underlying marked smooth liftings. Its reduction is a group isomorphism. Replacing h by $h - h(e_{\widetilde{A}})$ makes it a group isomorphism without changing its reduction over R. It consequently respects the markings over k and gives the required isomorphism of marked abelian schemes.

For surjectivity, a smooth scheme lifting of A is proper, since properness is invariant under nilpotent thickenings. Smoothness lifts its identity section to $\widetilde{R}$. Proposition 4.35 below then gives an abelian scheme structure with this lifted identity. The given marking over R is a group isomorphism because it preserves the identities. Different choices of the lifted identity give isomorphic marked abelian schemes by translation. □

Proposition 4.35. *Let $p : \widetilde{R} \to R$ be a small surjection in $\mathcal{C}$ with kernel I. Put $S = \operatorname{Spec} R$ and $\widetilde{S} = \operatorname{Spec} \widetilde{R}$. Let $\widetilde{\pi} : \widetilde{A} \to \widetilde{S}$ be proper and smooth, with a section $\widetilde{e}$. Suppose its reduction A/S is an abelian scheme with identity $e = \widetilde{e}|_S$. Then $\widetilde{A}$ admits a unique abelian scheme structure with identity $\widetilde{e}$ extending that of A.*

Proof. Let $g : A \times_S A \to A$ be subtraction, $g(x, y) = x - y$. Our first task is to lift it to a morphism $\widetilde{g} : \widetilde{A} \times_{\widetilde{S}} \widetilde{A} \to \widetilde{A}$. Set $A_0 = A \otimes_R k$ and $g_0 = g \otimes_R k$. Its obstruction lies in

$$o_0 \in H^1(A_0 \times_k A_0, g_0^* \underline{\mathrm{Der}}_{A_0/k}) \otimes_k I.$$

Consider the two morphisms

$$j_1(x) = (x, \widetilde{e}), \qquad j_2(x) = (x, x)$$

from $\widetilde{A}$ to $\widetilde{A} \times_{\widetilde{S}} \widetilde{A}$, and use the same notation for their reductions. Naturality of the obstruction gives

$$j_i^* o_0 = o(g \circ j_i; \widetilde{A}, \widetilde{A}; \widetilde{R} \to R) = 0.$$

Indeed, $g \circ j_1 = \mathrm{Id}_A$ lifts to $\mathrm{Id}_{\widetilde{A}}$, and $g \circ j_2 = e \circ \pi$ lifts to $\widetilde{e} \circ \widetilde{\pi}$.

Invariant vector fields trivialize the tangent sheaf of A_0:

$$\underline{\mathrm{Der}}_{A_0/k} \simeq \mathcal{O}_{A_0} \otimes_k \mathrm{Lie}_{A_0/k}\,.$$

Since $H^0(A_0, \mathcal{O}_{A_0}) = k$, the Künneth formula yields

$$\begin{aligned} &H^1(A_0 \times_k A_0, g_0^* \underline{\mathrm{Der}}_{A_0/k}) \otimes_k I \\ &\quad \simeq \left(\mathrm{pr}_1^* H^1(A_0, \mathcal{O}_{A_0}) \oplus \mathrm{pr}_2^* H^1(A_0, \mathcal{O}_{A_0})\right) \otimes_k \mathrm{Lie}_{A_0/k} \otimes_k I. \end{aligned}$$

Writing $o_0 = (u, v)$ in these two summands, we have $j_1^* o_0 = u$ and $j_2^* o_0 = u + v$. Thus $u = v = 0$, so a lift $\widetilde{g}$ exists.

The set of lifts is a torsor under

$$H^0(A_0 \times_k A_0, g_0^* \underline{\mathrm{Der}}_{A_0/k}) \otimes_k I \simeq \mathrm{Lie}_{A_0/k} \otimes_k I.$$

The possible values of a lift at $(\widetilde{e}, \widetilde{e})$ form a torsor under the same group. Evaluation induces the identity on these two groups, so there is a unique lift satisfying

$$\widetilde{g}(\widetilde{e}, \widetilde{e}) = \widetilde{e}.$$

Define

$$\widetilde{\iota}(y) := \widetilde{g}(\widetilde{e}, y), \qquad \widetilde{m}(x, y) := \widetilde{g}(x, \widetilde{\iota}(y)).$$

These reduce to inversion and addition on A and preserve the chosen basepoints.

To verify the group identities, note that any two morphisms $h_1, h_2 : \widetilde{A}^r \to \widetilde{A}$ with the same reduction differ by an element of

$$H^0(A_0^r, h_0^* \underline{\mathrm{Der}}_{A_0/k}) \otimes_k I \simeq \mathrm{Lie}_{A_0/k} \otimes_k I,$$

where h_0 is their common reduction. Evaluation at $(e_0, \ldots, e_0)$ is an isomorphism on this group. Hence $h_1 = h_2$ whenever both send $(\widetilde{e}, \ldots, \widetilde{e})$ to $\widetilde{e}$. Apply this to the two sides of associativity, the identity laws, the inverse laws, and commutativity. Each identity holds over S and at the chosen basepoint, so it holds over $\widetilde{S}$.

The geometric fibers are unchanged by the nilpotent thickening and are connected. Thus the resulting proper smooth group scheme is an abelian scheme. Any other extending group law has a subtraction map with the same normalization, so uniqueness of $\widetilde{g}$ proves uniqueness of the group structure. □

Corollary 4.36. *Let $S \hookrightarrow \widetilde{S}$ be a closed immersion defined by a nilpotent ideal sheaf. Let $\widetilde{A}/\widetilde{S}$ be an abelian scheme with reduction A/S, and let $\lambda : A \to A^\vee$ be a polarization. Any homomorphism $\widetilde{\lambda} : \widetilde{A} \to \widetilde{A}^\vee$ restricting to λ is a polarization of $\widetilde{A}$.*

The homomorphisms $\widetilde{\lambda}$ and $\widetilde{\lambda}^\vee$ have the same reduction, so rigidity gives $\widetilde{\lambda} = \widetilde{\lambda}^\vee$ under the canonical bidual identification. Its restriction to every geometric fiber is a polarization because the geometric fibers of S and $\widetilde{S}$ are the same. The fiberwise criterion for a symmetric homomorphism to be a polarization therefore applies.

5. Algebraic Spaces and Algebraic Stacks

This section recalls the definitions needed for moduli problems; see [1, Appendix A]. The categorical conventions are recalled in the appendix below.

5.1. **Spaces and algebraic spaces.** We use the étale topology on schemes over a fixed scheme S. A covering is a family of étale morphisms whose images jointly cover the target. Thus the covering maps need not be open immersions.

Definition 5.1. The big étale site of S is the category Sch/S endowed with these covering families.

Definition 5.2. A *space* over S is a sheaf of sets on the big étale site of S. We write Spc/S for the category of such sheaves, with natural transformations as morphisms.

Remark 5.3. Every scheme X/S defines a space by its functor of points

$$U \longmapsto X(U) = \mathrm{Hom}_S(U, X).$$

Indeed, morphisms to a scheme satisfy étale descent.

Warning 5.4. Effective descent for schemes does not hold for arbitrary étale descent data. The resulting quotient sheaf may be an algebraic space without being a scheme.

Remark 5.5. Yoneda's lemma identifies Sch/S with a full subcategory of Spc/S.

Definition 5.6. An *algebraic space* X/S is a space such that:

(1) : the diagonal $\Delta_X : X \to X \times_S X$ is representable by schemes;

(2) : there is a scheme U and a representable, surjective étale morphism $U \to X$.

Here a morphism $F \to G$ of spaces is representable by schemes if $T \times_G F$ is a scheme for every scheme T mapping to G. The morphism $U \to X$ is called an *étale atlas*.

For such an atlas, $R := U \times_X U$ is a scheme, the projections $R \rightrightarrows U$ are étale, and X is the quotient sheaf U/R. This allows one to study algebraic spaces locally in the étale topology using schemes.

5.2. **Stacks and algebraic stacks.**

Definition 5.7. A *stack* F over S is a contravariant pseudofunctor from Sch/S to the category of groupoids satisfying descent for the étale topology. Explicitly, let $\{V_i \to U\}_{i\in I}$ be an étale covering, and put $V_{ij} = V_i \times_U V_j$ and $V_{ijk} = V_i \times_U V_j \times_U V_k$.

(1) : For $x, y \in \operatorname{Ob} F(U)$, the presheaf

$$(V \to U) \longmapsto \operatorname{Isom}_{F(V)}(x|_V, y|_V)$$

is a sheaf. In particular, compatible local isomorphisms glue uniquely.

(2) : If $x_i \in F(V_i)$ and isomorphisms $\varphi_{ij} : x_i|_{V_{ij}} \xrightarrow{\sim} x_j|_{V_{ij}}$ satisfy $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ on V_{ijk}, then there are $x \in F(U)$ and isomorphisms $\varphi_i : x|_{V_i} \xrightarrow{\sim} x_i$ with $\varphi_j = \varphi_{ij} \circ \varphi_i$ on V_{ij}.

Pullbacks in a pseudofunctor are compatible with composition up to specified coherent isomorphisms. Equivalently, one may use a category fibered in groupoids over Sch/S satisfying these two descent conditions.

Definition 5.8. An *algebraic stack* $\mathcal{X}/S$ is a stack whose diagonal $\mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is representable by algebraic spaces and which admits a smooth surjective atlas $U \to \mathcal{X}$ from a scheme. If an étale surjective atlas exists, $\mathcal{X}$ is a *Deligne–Mumford stack*.

An algebraic space is viewed as a stack by making each set of sections a groupoid with only identity arrows. Stacks also retain the automorphisms of objects, which are essential in moduli problems without sufficiently rigid level structure.

6. Appendix

6.1. **Grothendieck Topology and the étale topology.** A Grothendieck topology specifies covering families compatible with isomorphisms, base change, and composition of coverings. For a site with fiber products, a presheaf of sets F is a sheaf if, for every covering $\{U_i \to U\}$, the diagram

$$F(U) \longrightarrow \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \times_U U_j)$$

is an equalizer. Thus compatible local sections glue uniquely.

In the étale topology, coverings are jointly surjective families of étale morphisms. The small étale site $X_{\text{ét}}$ uses only schemes étale over X, whereas the big site allows all schemes over X. The fppf topology instead uses jointly surjective families of flat morphisms locally of finite presentation. Relative Picard functors and quotients by finite flat group schemes are naturally considered in the fppf topology.

6.2. **Category theory.**

6.2.1. *2-categories and 2-functors.*

Definition 6.1. A strict 2-category $\mathcal{C}$ consists of the following data:

(1) : a collection of objects $\operatorname{Ob}\mathcal{C}$;

(2) : for each pair X, Y, a category $\operatorname{Hom}_{\mathcal{C}}(X, Y)$, whose objects are called 1-morphisms and whose arrows are called 2-morphisms;

(3) : composition functors

$$\operatorname{Hom}_{\mathcal{C}}(Y, Z) \times \operatorname{Hom}_{\mathcal{C}}(X, Y) \longrightarrow \operatorname{Hom}_{\mathcal{C}}(X, Z), \qquad (g, f) \longmapsto g \circ f,$$

and identity 1-morphisms Id_X, satisfying strict associativity and identity laws. Functoriality of composition gives the interchange law between horizontal and vertical composition of 2-morphisms.

A strict 2-functor preserves these data and their compositions. A pseudofunctor preserves composition and identities up to coherent invertible 2-morphisms. Categories, functors, and natural transformations form the basic example of a 2-category. All collections are taken in fixed universes so that the required functor categories are defined.

6.3. **The "quasi"s.**

Definition 6.2. Let $f : X \to Y$ be a morphism of schemes.

It is **quasi-compact** if $f^{-1}(V)$ is quasi-compact for every affine open $V \subseteq Y$.

It is **quasi-separated** if the diagonal $\Delta_f : X \to X \times_Y X$ is quasi-compact.

It is **quasi-finite** if it is of finite type and every fiber is a finite scheme over its residue field, equivalently, has finitely many points.

A scheme X is **quasi-affine** if it is an open subscheme of an affine scheme. A morphism f is quasi-affine if $f^{-1}(V)$ is quasi-affine for every affine open $V \subseteq Y$.

6.4. **Cohomology theories.**

6.4.1. *Cup product.* Let Z be a quasi-compact separated scheme and let $\mathcal{F}, \mathcal{G}$ be quasi-coherent $\mathcal{O}_Z$-modules. The cup product is a natural pairing

$$H^i(Z, \mathcal{F}) \otimes H^j(Z, \mathcal{G}) \longrightarrow H^{i+j}(Z, \mathcal{F} \otimes_{\mathcal{O}_Z} \mathcal{G}).$$

Choose a finite affine open covering $\mathcal{U} = \{Z_\alpha\}$. Its finite intersections are affine, so its Čech complexes compute these cohomology groups. Write $Z_{\alpha_0 \cdots \alpha_r} = \bigcap_{t=0}^{r} Z_{\alpha_t}$. For cochains $a \in C^i(\mathcal{U}, \mathcal{F})$ and $b \in C^j(\mathcal{U}, \mathcal{G})$, set

$$(a \smile b)_{\alpha_0 \cdots \alpha_{i+j}} = a_{\alpha_0 \cdots \alpha_i}|_{Z_{\alpha_0 \cdots \alpha_{i+j}}} \otimes b_{\alpha_i \cdots \alpha_{i+j}}|_{Z_{\alpha_0 \cdots \alpha_{i+j}}}.$$

With the differential

$$(\partial a)_{\alpha_0 \cdots \alpha_{i+1}} = \sum_{t=0}^{i+1} (-1)^t a_{\alpha_0 \cdots \widehat{\alpha_t} \cdots \alpha_{i+1}}|_{Z_{\alpha_0 \cdots \alpha_{i+1}}},$$

one checks

$$\partial(a \smile b) = (\partial a) \smile b + (-1)^i a \smile (\partial b).$$

Consequently $[a] \smile [b] = [a \smile b]$ is well defined on cohomology.

6.5. **Upper semi-continuity.**

Theorem 6.3. *Let $f : X \to Y$ be a proper morphism of Noetherian schemes. The function $y \mapsto \dim X_y$ is upper semi-continuous, with $\dim \varnothing = -\infty$. Equivalently,*

$$\{y \in Y : \dim X_y \geq n\}$$

is closed for every integer n.

Proof. For a finite type morphism, the set $Z_n = \{x \in X : \dim_x X_{f(x)} \geq n\}$ is closed. Properness implies that $f(Z_n)$ is closed, and $f(Z_n) = \{y : \dim X_y \geq n\}$. See [5, Tag 0D4I]. □

Properness is necessary here: the open immersion $\mathbb{A}^1_k \setminus \{0\} \hookrightarrow \mathbb{A}^1_k$ has zero-dimensional fibers away from 0 and an empty fiber at 0, so the corresponding set for $n = 0$ is not closed.

For cohomology of fibers, the corresponding result is the following; see [2, III, Thm. 12.8] in the projective case and [5, Tag 07VJ] in general.

Theorem 6.4 (Grothendieck)**.** *Let $f : X \to S$ be a proper morphism of Noetherian schemes, and let $\mathcal{F}$ be a coherent sheaf on X, flat over S. Write $\mathcal{F}_s$ for its pullback to X_s. For every $i \geq 0$, the function*

$$s \longmapsto \dim_{k(s)} H^i(X_s, \mathcal{F}_s)$$

is upper semi-continuous.

Proof. We first recall flat base change, which will also be used elsewhere.

Lemma 6.5. *Let $f : X \to Y$ be a separated morphism of finite type between Noetherian schemes, let $\mathcal{F}$ be quasi-coherent on X, and let $u : Y' \to Y$ be a flat morphism of Noetherian schemes. In the Cartesian square*

$$\begin{array}{ccc} X' & \xrightarrow{v} & X \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{u} & Y, \end{array}$$

there are natural isomorphisms $u^*R^i f_*\mathcal{F} \simeq R^i g_* v^*\mathcal{F}$ for all $i \geq 0$.

Proof of the lemma. The assertion is local on Y and Y', so take $Y = \operatorname{Spec} A$ and $Y' = \operatorname{Spec} A'$. A finite affine covering of X and its pullback to X' have affine intersections. Their Čech complexes satisfy

$$C^\bullet(\mathcal{U}', v^*\mathcal{F}) \simeq C^\bullet(\mathcal{U}, \mathcal{F}) \otimes_A A'.$$

Flatness makes tensoring exact, so it commutes with cohomology. Hence

$$H^i(X', v^*\mathcal{F}) \simeq H^i(X, \mathcal{F}) \otimes_A A'.$$

□

To prove semicontinuity, work locally with $S = \operatorname{Spec} A$. The finite-complex theorem for proper morphisms gives, locally on S, a bounded complex $K^\bullet$ of finite free A-modules such that, for every s,

$$H^i(X_s, \mathcal{F}_s) = H^i(K^\bullet \otimes_A k(s)).$$

This uses flatness of $\mathcal{F}$ over S and also computes cohomology after arbitrary base change; see [5, Tag 07VJ]. Writing $d^i : K^i \to K^{i+1}$, we obtain

$$\dim_{k(s)} H^i(K^\bullet \otimes_A k(s)) = \operatorname{rank} K^i - \operatorname{rank} d^{i-1}(s) - \operatorname{rank} d^i(s).$$

Matrix rank is lower semi-continuous, since the condition that a minor is nonzero is open. The displayed dimension is therefore upper semi-continuous. □

6.6. **Künneth Formula.**

Theorem 6.6 (Künneth formula)**.** *Let k be a field, and let X and Y be quasi-compact k-schemes with affine diagonals, for example quasi-compact separated k-schemes. For quasi-coherent sheaves $\mathcal{F}$ on X and $\mathcal{G}$ on Y, pullback and cup product induce a canonical isomorphism*

$$\bigoplus_{p+q=n} H^p(X, \mathcal{F}) \otimes_k H^q(Y, \mathcal{G}) \xrightarrow{\sim} H^n(X \times_k Y, \operatorname{pr}_1^*\mathcal{F} \otimes_{\mathcal{O}_{X\times_k Y}} \operatorname{pr}_2^*\mathcal{G}).$$

One may compute this map using the two affine Čech coverings and their double complex; over a field the algebraic Künneth formula has no higher Tor terms. See [5, Tag 0BEC].

6.7. **Reducing to a field.** Some properties of morphisms can be checked on geometric fibers; others require additional hypotheses on the family.

Proposition 6.7. *Let $f : X \to S$ be a morphism, and let P be a property stable under base change and detectable on geometric fibers. Then f has P if and only if $X_{\bar{s}} \to \operatorname{Spec} \overline{k(s)}$ has P for every geometric point $\bar{s} \to S$.*

For example, if f is flat and locally of finite presentation, then it is smooth if and only if all geometric fibers are smooth. Fiberwise smoothness alone does not imply smoothness of f: the closed immersion $\operatorname{Spec} k \to \operatorname{Spec} k[\epsilon]/(\epsilon^2)$ has a smooth geometric fiber but is not flat.

Over a field, properties that descend under faithfully flat field extension may be checked after extending to an algebraic closure. For statements about morphisms of sheaves or about infinitesimal structure, checking values at geometric points alone is generally insufficient; flatness, base change, or an appropriate descent theorem must also be used.

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